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MATHESIS ENUCLEATA:

OR, THE

ELEMENTS

OF THE

MATHEMATICKS.

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The AUTHOR's PREFACE TO THE READER,

Containing a SYNOPSIS of his Method.

I.

THAT the Reader may the better apprehend our deſign and aim, we have thought fit to premiſe ſome things concerning the Methods, both general and particular, we make uſe of in the following Treatiſe. For as heretofore a ſort of a blind deference to, and ſuperſtitious Veneration of Antiquity, and eſpecially of Ariſtotle, *has hindred the growth and progreſs of Natural Philoſophy, which of late has made ſuch conſiderable advances, ſince it has ventured to ſtand upon its own Bottom, to make new Additions to former Inventions, to eſſay new and unknown Objects, to ſubſtitute Things inſtead of Names, Certainties inſtead of Doubts, and Experience in the room of dull Credulities; not derogating in the mean while from the deſerved Praiſes of the Ingenious among the Ancients* [illegible] *without doubt Mathematicks alſo, unleſs our Predeceſſors had imagined that it had long ago been brought to its utmoſt Perfection by* Euclid, Archimedes, Apollonius, *and other Ingenious Ancients, would have arrived long ſince to a higher pitch, and by this time have ſurpaſs'd thoſe Limits, which now we admire its arrival to.*

II.

It is confeſs'd by all, that no Human Knowledge whatſoever can lay a more juſt claim to an unſhaken Evidence and Certainty, or boaſt an higher neceſſity of its Demonſtrations, or a greater multitude of undeniable Truths, than the Mathematicks; and that thoſe Propoſitions we have, found out by Archimedes, *demonſtrated by* Euclid, Apollonius, *and others, are at the ſame time unqueſtionable and altogether wonderful. But we may with Truth affirm, that moſt of their Propoſitions may either be diſpoſed in a better order, or propounded eaſier, or demonſtrated more evidently and directly, or taught after a more ſhort and compendious way, now at leaſt after they are already found out, and with a great deal of Pains demonſtrated by their firſt Inventors; and of this Opinion are ſeveral of the beſt and moſt celebrated Mathematicians of the preſent Age.*

III.

It is certain Euclid *has demonſtrated ſeveral Propoſitions, (as* Prop. 2, 3, 20, 30, lib. 1. 2, 5, 6, 10, 15, 28, 29, lib. 3. *&c.) whoſe Truth to any attentive Perſon appears from the very terms, more clearly and certainly than the truth of* Axiom 13. lib. 1. *which his Interpreters dare not admit without a Demonſtration. And tho' thoſe ſuperfluous Demonſtrations derogate nothing from the certitude of the thing, yet by an unneceſſary increaſe of the number of Propoſitions, and (which frequently follows thence) an inverting the order of things, they breed Tedioufneſs and Confuſion.*

IV.

There are none, unleſs thoſe who are bigotted to Antiquity, but muſt own that the Elements of Euclid *are deſtitute of a juſt and orderly Diſpoſition of things. For to omit, that in the firſt Book there are handled ſeveral ſorts of Subjects, and a great variety of Properties demonſtrated of them promiſcuouſly without any reſpect to ſimilitude or conveniency; there is this never to be* *excuſed, that, after he has in the firſt Book deduced and demonſtrated ſome particular Affections of Magnitude, he proceeds, in the ſecond, to thoſe things which are univerſal and common to any quantity; then in the third and fourth, he contemplates the Circle and the Properties of Figures inſcrib'd in it, or circumſcribed about it; in the fifth again he treats of the univerſal Doctrine of Reaſons and Proportions; and yet not ſo univerſally, but in the ſeventh again he is obliged to demonſtrate the ſame of Numbers particularly, which might have been done for all Quantities whatſoever, by one general Demonſtration.*

V.

Then as for the method of Demonſtrating uſed by the Ancients, it is true that it nicely regarded the certainty of its Concluſions, nor would it admit any thing into its Demonſtrations, which was not either a firſt Principle and ſo ſelf evident, (called by them an Axiom*) or might not be ſuppoſed, beyond all Contradiction, poſſible to be effected (and on that account named a* Poſtulate*) or, thirdly, an arbitrary Denomination of the thing propoſed which needed no Demonſtration, (and was called a Definition or Explication of a Term;) or, laſtly, which had not been evidently demonſtrated before: Yet I believe none will deny, but that this Method would have been more deſervedly eſteem'd, if with the certainty of its Concluſions it had joined a greater Eaſineſs, Brevity, and Evidence, which is wanting in moſt of the Demonſtrations of the Ancients; who thought it enough, firmly and infallibly to eſtabliſh the truth of their Theorems, and extort the Aſſent of their Readers; little regarding by what Ambages, by how many circumambulatory Propoſitions, and almoſt whole Volumes, it was done; that thereby they might be forced to acknowledge the thing to be ſo, while how it came to be ſo, or from what intrinſick Cauſe or Condition of the Subject requiring it, ſuch and ſuch an Attribute agreed to it, remained in the mean while obſcure, or altogether unknown.*

VI.

Hence they made ſuch frequent uſe of Apagogical Demonſtrations, or deductive ad abſurdum & impoſſibile, *which ought not to be done, but where no oſtenſive Demonſtration can be had, or for illuſtrating negative Propoſitions rather than demoſtrating them; for the method of Deduction* ad impoſſibile, *does not ſo much demonſtrate the Truth it ſelf directly, as the conſequent Abſurdity of the oppoſite Suppoſition; whence it follows very indirectly, (tho' moſt certainly) that the Propoſition is true, while in the mean time the original Reaſon of its Truth remains altogether hid and in the dark.*

VII.

But that we may not ſeem unjuſtly to reject the particular Method of the Ancients, made uſe of by Euclid, *as in* lib. 12. Prop. 2, 10. &c. *and by others, but eſpecially by* Archimedes, *who peculiarly addicted himſelf to it, whence it has been by ſome called the* Archimedean *Method, and by* Renaldinus *the Method* per Exploſum exceſſum atque defectum; *beſides its Deduction doubly* ad abſurdum, *whereon it always relies, as* e. g. *it infers the equality of two Magnitudes* A *and* B *by a far fetch round-about way, by ſhewing, that if* B *be ſuppoſed greater or leſs than* A, *from either Poſition there would follow an abſurdity; and thence as it were begging the Equality by a new Inference, which tho' it may paſs free from Suſpicion, yet it neither ought, nor can be admitted, without this Limitation: In comparing thoſe things whoſe Natures are capable of Equality, if it can be demonſtrated that the one is neither greater nor leſs than the other, we may thence juſtly infer their Equality.*

VIII.

The Learned are now generally agreed, that, beſides that Synthetick Method, whereby the Ancients either oſtenſively deduced their Problems and Theorems from evident and common Principles,

ples, or apagogically demonſtrated them by Deductions ad abſurdum, *they alſo made uſe of a certain ſort of Analyſis, whereby they found out thoſe Theorems and Problems; and which, to raiſe the greater Admiration in their Readers, they afterwards ſtudiouſly conceal'd and kept to themſelves: Which Method is undoubtedly preferable to the other, as not only demonſtrating the certainty of the Propoſitions ſo found, but at the ſame time ſhewing the invention of them too; and this is that Method that* Vieta, Harriot, *and* Des Cartes, *and their Followers have not only brought to light in this laſt Age, but to a great degree of Perfection too, and whereof* Carolus Renaldinus, *in that vaſt Work he has intituled* Ars Analytica Mathematum, *has given us a large Treatiſe.*

IX.

*There has appear'd moreover of late another particular Method * invented by* Bonaventura Cavallerius, *which is called* the Method of indiviſibles, *whereby the moſt difficult and abſtruſe Problems of Geometry are found out and Demonſtrated with an incredible eaſe, which is the above-mentioned* Renaldinus's *Opinion of it*, lib. 1. Reſol. & Comp. p. 239. *which, to demonſtrate the Equality or Proportions of Figures and Bodies that may be compared with one another, goes to work after a way which ſeems to be more natural than any other, by ſuppoſing plane Figures to conſiſt of innumerable lines, and ſolids of innumerable Plans (called their indiviſible Parts or Elements, becauſe the Lines are conceived without latitude, and the Plans without any thickneſs,) and relying on this ſelf-evident* Axiom, *That if all the Indiviſibles of one Magnitude collectively taken, be equal or proportional to all the correſpondent Indiviſibles of another, or taken ſeparately each to each, then alſo thoſe Magnitudes will be equal or proportional among themſelves. Which Inference can be guilty of no Fallacy, nor liable to any Error, as long as thoſe Elements are taken and conceived in that ſenſe their Authors deſign them; which is ſufficiently demonſtrated by* Renaldinus, lib. 1. Compoſ. & Reſol. p. 245. *towards the bottom,*

* Dr. *Wallis* is of Opinion this is nothing but an Improvement of the Ancients Method of Exhauſtions.

tom, and at the beginning of p. 306. *and alſo by* Honoratus Faber, *in his Synopſis,* p. 24. *and* Dr. Barrow *in* Lect. Geom. p. 24. *and the following, againſt* Tacquet *and the other Adverſaries of this method.*

X.

There is another Method a-kin to this, which may be properly named Generative, very much followed by Faber *in his Synopſis, and* Barrow *in his* Lect. Geom. *whoſe Author* Renaldinus *tells us was* Guldinus, lib. Cit. p. 253. *ſhewing at large its Rules and Foundations in the following pages,* viz. *The riſe of Lines from certain motions of points; of Plane and Curvilinear Surfaces, from the determinate progreſs or rotation of a given Line; and of Solids by the various motions of various Surfaces; the Productions whereof are ſo repreſented to the Imagination, that the intrinſick nature of the magnitudes thence ariſing may become known, and their Properties and Affections may, from their Natures thus known, be eaſily and briefly deduced.*

XI.

Near a-kin to this Method of Cavellerius *is that other of* Infinite Progreſſions, *wherein having found a certain Progreſſion of like Parts circumſcrib'd about, or inſcrib'd in any given magnitude, which may be continued by Biſection* ad infinitum; *and then at length (by vertue of the Doctrine of Exhauſtions, founded on* Prop. 1. lib. 10. Eucl.*) will terminate in the magnitude it ſelf, I ſay, wherein the ſum of thoſe infinite Terms, collected by Rules on purpoſe for the addition of thoſe Progreſſions, and conſequently the quantity or proportion of the propoſed Magnitude, to any other given one, may be expreſſed or defined. But this termination of Figures infinitely circumſcribed or inſcribed in a Circle, not pleaſing* Renaldinus *(altho' his Diſſention ſeems only to conſiſt in Words,) he exhibits another Method like it, (which he peculiarly calls his own) built on twelve fundamental Theorems, and illuſtrated by ſeveral Examples,* lib. 1. de Reſol. & Comp. p. 277 & ſeqq.

XII.

Of late alſo, the moſt ingenious Mr. Iſaac Newton, *to demonſtrate his* Philoſophiæ Naturalis Principia Mathematica, lib. 1. ſect. 1. *premiſes ſome Lemma's of his method of* Rationes primæ & ultimæ, *or evaneſcent quantities, thereby to avoid the tediousneſs of deducing long and perplext apagogical Demonſtrations after the way of the Ancients. For finding that his Demonſtrations might be very much contracted by the method of Indiviſibles, and knowing at the ſame time, that that method was ſcrupled by ſome, and thought not very Geometrical, he rather choſe to found his Method on the ſums and proportions of quantities which he calls Evaneſcent, which performs the ſame as the Method of Indiviſibles, and may be more ſafely uſed, which he inculcates in theſe very words, and others ſuch like, in* Schol. *of* Lemma. 11. *And alſo anſwers ſeveral objections which might ſeem to make againſt it.*

XIII.

But it would be in vain for us to attempt, in this place, to explain all and each of thoſe various methods at length; having only propoſed to our ſelves, to demonſtrate the chief and principal Theorems and Inventions of the Mathematicks, and to uſe ſometimes one of them and ſometimes another, (having firſt Demonſtrated their Foundations) according as we ſhall judge this or that of them, fitteſt to Demonſtrate the thing in hand, and ſo ſhew the reaſons, and uſe of each of them in the proceſs of this diſcourſe. And altho' H. Faber *in his* Synopſis, p. 8. *Inſinuates, that Analytick terms ought not to be made uſe of in Geometrical Demonſtrations, becauſe that Algebraick method ſeems to be too difficult for young beginners; yet we are of the quite contrary opinion (nay we can ſcarce doubt but that that Ingenious man would alſo agree with us herein, if he ſaw the way we make uſe of thoſe foundations of Algebra, which is only of the moſt ſimple and general principles of it) eſpecially in this caſe, where the ſaid method is by little and little inſtill'd with the Demonſtrations themſelves; and the literal Computations taught from their firſt*

Principles, than which nothing is more eaſie: And this is that which we deſign to do, and ſo uſe the learner by degrees to this ſort of Demonſtration, thereby to prepare him the better for the Analytick Geometry of the Moderns, which is the higheſt apex *of the Mathematicks. But we had rather our Reader ſhould himſelf find, than we trouble our ſelves any further to tell him here, how compendiouſly we Demonſtrate the Propoſitions of Geometry, by the help of theſe Analytick notes, without the tedious Concatenation of a long Chain of Conſequences, which would be otherwiſe unavoidable.*

XIV.

After this way we deſign to go through the following Scheme. 1. *We ſhall deduce ſeveral propoſitions of* Euclid, Archimedes, *and* Apollonius *from our definitions, and the generations of Magnitudes therein propoſed; as Corollaries neceſſarily flowing from them, and confirm'd only by an immediate and ſimple conſequence.* 2. *We ſhall demonſtrate their chief Theorems (for the ſake of which they were forced to Demonſtrate ſeveral others before-hand, the knowledge whereof for their own ſake was not ſo neceſſary or valuable) without any long ſeries of antecedent Propoſitions, or Foreign principles, from a few direct and intrinſick Principles of their own. Whence* 3. *It will follow, that after this Method we ſhall propoſe things, and treat of them, in a more natural Order, and firſt of all deliver thoſe which are moſt univerſal and common to all quantities, and then deſcend to thoſe which in a more ſpecial manner regard Magnitude; and diſtribute and diſpoſe all according to certain general diſtinct Claſſes of the things to be treated of, and their affections. Hence alſo* 4. *We deduce from thoſe univerſal Theorems, by way of Corollary, the Precepts of vulgar Arithmetick, and ſpecious Computation, which afterwards we make uſe of in particular Demonſtrations after a very ſhort and compendious way; and, for this very reaſon, ſome Learned men of the preſent Age are of opinion, that the Ancients often fell into that tedious and intricate prolixity in their Demonſtrations, becauſe they would not acknowledge the great affinity there was between Arithme-*

tick and Geometry, taking particular care not to introduce the Terms and Operations of Arithmetick into Geometry; tho' at the ſame time they never ſcrupled to transfer the names of Plan, Square, Cube, and ſuch like to numbers. 5. *Laſtly, Having firſt Demonſtrated the firſt and Fundamental Theorems of Elementary Geometry, we may ſafely build on them the* Praxes *of all kinds of Mathematical Arts, that are moſt uſeful and requiſite to ſeveral Exigencies of human Life, as, firſt,* Trigonometry *both Plain and Spherical, the Conſtruction and uſe of the Tables of Sines and Tangents.* 2. *The* Conſtruction *of* Logarithms, *and a compendious application of them to* Trigonometry: *And in the* 3. *and laſt place, the fundamental Precepts of Algebra, or the Analytic Art; by the help whereof the learner may at length arrive to the higher and more recluſe parts of Geometry, and become maſter thereof: Not to mention ſeveral Geometrical and Arithmetical Problems, which we have all along derived from ſeveral of our Theorems, by way of Corollary, which it may be ſome other time, may make an Appendix of this work.*

XV.

And thus when we ſhall have Demonſtrated not only the chief Theorems of the Ancient Mathematicians, omitting the unneceſſary crowd of thoſe that are only Subſidiary, but alſo have demonſtratively deduc'd the fundamental Precepts of the moſt neceſſary and uſeful Arts that flow from them, and that are abſtracted from matter, as of Arithmetick, Trigonometry and Algebra; I hope none will doubt but that in this little Volum we have exhibited, as it were the Nucleus *or Kernel of the pure and genuine Mathematicks' (for thoſe other Sciences and Arts which go by the name of mixt Mathematick, are moſt of them parts of natural Philoſophy, from the application of Mathematicks to the Phænomena of nature) and ſo may juſtly bear the name of* Matheſis Enucleata.

XVI.

Nor are we ignorant, nor ſhall we conceal what ſeveral Learned men have both propoſed and already done, for removing thoſe difficulties and blemiſhes of the Ancient Mathematical Methods we have juſt now mentioned. The late Admoniſhments of the anonymous Authour of L'Art de Penſer, *no leſs ingeniouſly than modeſtly delivered*, Part. 4. Chap 9. 10. *of the ſaid Treatiſe, are ſufficiently known; as alſo the laudable endeavours of* A. Tacquet *and* Honoratus Faber *and ſeveral others above mentioned for contracting, new ordering, and more eaſily and directly Demonſtrating the chief Geometrical Inventions of the Ancients. There are moreover extant of a certain anonymous Author*, Elementa Geometrica novo ordine ac methodo feré Demonſtrata, *Printed at* London *about* 26. *Years ago There are alſo* F. Ignatius Gaſton Pardies Elemens de La Geometrie, *&c. Tranſlated into Latin after the third Edition, by the Famous* Schmidtius *Profeſſor at* Geneva: *As alſo of* F. Mich. Mourgues s, *of the Society of* J. Nouveaux Elemens de Geometrie, abreges par des Methodes particulieres en moins de Cinquante propoſitions, *&c. There are alſo ſeveral other Eſſays of reducing the Mathematicks into a better Order and Method, the titles whereof we have only as yet ſeen; and even while theſe papers were in the Preſs, there happen'd into our hands a Treatiſe of* F. Lamy's *Entituled* Les Elemens de Geometrie, ou de la meſure des Corps, *&c. Printed at* Paris *in* 16 5 : *ſo that we may only ſeem to ſome to do what has been done already, in endeavouring to ſhew our Reader a new and ſhorter way to the Mathematicks.*

XVII.

But as none can blame Jacobus le Maire, *becauſe, after the happy diſcovery of the* Magellanick *Paſſage from the* Atlantick *into the* Pacifick *Sea, he would needs yet endeavour to find another ſhorter, which he accordingly did; nor can they be blamed, who now a days conſult about finding one from theſe parts of* *the World, by the North to the* Eaſt India's. *Thus alſo, in an affair of that moment, that one or a few are not ſufficient to bring it to Perfection, if any one who comes after, not only invited, but alſo aſſiſted by the ingenious Endeavours of thoſe who have gone before him, ſhall undertake to add to their Inventions, to help on the buſineſs by his advices, and ſhew what things are capable of a further Poliſh, and the method how to perform it, doubtleſs ſuch an one ought not to be blamed, nor accuſed of arrogance, unleſs at the ſame time he endeavours to depretiate the eſſays of others, and cry up his own as the only valuable; Which how far it is from our deſign, the work it ſelf will abundantly ſhew. Moreover as the ſenſes of men are differently affected by different Objects, and their Palates have different Reliſhes of the ſame thing, according to different Preparations of them: So the ſame truth takes and inſinuates it ſelf more eaſily with one propoſed and demonſtrated after this way, more with another after that way; and we are ſo much the more likely to ſuit the different genii of different Perſons, by how many more and different methods and ways we ſhew them, leading to the ſame end, of which every one may take that which he likes beſt.*

XVIII.

We therefore Publiſh, by the Divine aſſiſtance, theſe our Endeavours alſo, after ſo many other ingenious and elaborate ones in the ſame kind; nor can we doubt the approbation of ſome of our Readers. This at leaſt we can experimentally affirm: That not a few of thoſe to whom theſe our thoughts were partly publickly read in Lectures, and partly privately taught (for they were only deſign'd for Learners) were not a little taken with the Conciſe *brevity and facility of the Demonſtrations; ſo that we may reaſonably hope to be acceptable to thoſe, to whom either time, or ſufficient force of genius is wanting, to run over the vaſt Volumes of the Ancient Mathematicians, and comprehend their prolix Demonſtrations, and long ſeries's of far fetch'd Conſequences; and as for thoſe who have both leiſure and genius to do ſo, this may ſerve for an Encouragement towards it; that after they have gone through the chief truths and propo-*

ſitions they contain, Demonſtrated in a more eaſie and ſhorter way, they may ſo much the more confidently adventure upon thoſe celebrated and ingenious Treatiſes, from the reading whereof they were before deterr'd by the length and almoſt inſuperable difficulty of their tedious and perplext Demonſtrations.

XIX.

Being now about to enter upon the matter it ſelf, we will only further hint theſe few things. 1. Since our whole deſign is for the advantage of young Students (which ought to be a Profeſſors chief Care and Study) we muſt not omit the Explication of the moſt ſimple terms; eſpecially ſince we deſign to deduce ſome Corollaries immediately from them, which heretofore have unneceſſarily increas'd the number of Propoſitions and Demonſtrations. 2. To encumber our work as little as we can with words we have made uſe, eſpecially in our Analytick Calculus, of ſome Symbols, as ⌶ *for Equality, as alſo of* □ *and* ▭ *for Square and Rectangle, and of* √ *the common Radicall ſign for the ſquare Root, with the Line* $\sqrt{}$ *on the top for connecting of quantities together, the Root whereof is jointly taken,* $\sqrt[3]{}$ *for the Cube Root,* $\sqrt{\sqrt{}}$ *for the Biquadratick Root.*

XX.

That the Reader may at one view ſee the Contents of the following Treatiſe, we have thought fit to preſent him here with it, by way of Synopſis. It is divided in two Books.

I.

The firſt whereof contains the chief and moſt ſelect Propoſitions of Euclid's *Elements, of* Archimedes's *Treatiſes of the Sphere and Cylinder, as alſo of the dimenſion of the* Circle, &c. *Wherein that which theſe Authors have demonſtrated by a long and tedious Series of Conſequences, and for the moſt part indirectly, we have here endeavoured to Demonſtrate directly, and ſo that the Demonſtration of each Propoſition depends, ei-*

ther

either on no other, and ſo is evident by its own light, or on a very few of the antecedent ones.

II.

After the ſame way in the ſecond Book we treat of the Conick Sections, *and Demonſtrate the chief properties of the* Conoid, Spheroid, Cycloid, Conchoid *and* Spiral Lines, *which are extant either in* Apollonius *or* Archimedes *and others, and what they have Demonſtrated by long and tedious proceſs's we have here exhibited in a ſhort and eaſie Compendium. And that,*

III.

In ſuch a method, as does not ſo much require intent and ſevere thinking, as a bare and eaſie inſpection, and application of the Principles of ſpecious Algebra, and method of Indiviſibles. which yet,

IV.

We don't barely ſuppoſe, and remit our Readers to other Books to learn (which would be too troubleſome) but in the Proceſs of the Work it ſelf, they are gradually, and as occaſion preſents, derived frrm their Original Fountain, and firſt Principles.

V

By the ſame way alſo the moſt uſeful and neceſſary Mathematical Praxes *are laid down under the names of* Corallaries *and* Scholia, *the Conſtruction of the Tables of Sines and Tangents taught, the Original and uſe of the Logarithms Demonſtrated, and the Precepts both of Plain and Spherical Trigonometry deduced from their firſt Principles,* &c.

VI.

The Praxis alſo of pure Arithmetick, and of common Decimal, and (which is ſeldom uſed) of Tetractical; as alſo the

Doctrine of Surds, are derived from their firſt Original. Whereunto,

VII.

As a Complement of the whole Work, we have added an Introduction to the Specious Analyſis, or new Geometry of the Moderns, particularly according to the Method of Des Cartes, *but much facilitated by later Inventions, and Comprizing the Precepts of the Art in ſix or ſeven Pages, but illuſtrated with above forty Examples in the different degrees of Equations.*

What the Reader's opinion will be of theſe our Endeavours, deſign'd only for the uſe of young Students, time muſt teach us. The Author himſelf at leaſt, amongſt his other performances, allows theſe the firſt place.

ADVERTISEMENT.

At the Hand and Pen in *Barbican* are Taught, *Viz.*

Writing, Arithmetick, Book-keeping, Algebra, Geometry, Meaſuring, Surveying, Gauging, Aſtronomy, Geography, Navigation and Dialing.

By *Robt. Arnold.*

Perſons Taught abroad.

Elementa Arithmeticæ Numeroſæ & Specioſæ. In uſum Juventutis Oxoniæ è Theatro Sheldoniano. Proſtant Venales Londini apud Dan Midwinter & Tho. Leigh *ad Inſigne Roſæ Coronatæ in Cæmeterio Divi Pauli.*

Matheſis Enucleata:

O R,

The Elements of the Mathematicks.

Book I.

Explaining the Firſt Principles of the Mathematicks; among which are (in the firſt place) *Definitions*, and ſome *Conſectaries* that flow from them.

CHAP. I.

Containing the Definitions or Explications of the Terms which relate to the Object of Mathematicks.

DEFINITION I.

MAthematicks is the Science or Knowledge of Quantity, and of Beings, as far as they are ſubject to it, or meaſurable; and may juſtly claim the Name of *Univerſal*, while it is employ'd in Demonſtrating thoſe Properties which are common to all or moſt *Quantities:* But when it deſcends to the different Species of Quantity, and is buſied in contemplating the Affections belonging particularly to this or that Quantity, it is diſtinguiſhed by various Names, and diſtributed into various Parts, according to the various diverſities of the Objects.

DEFINITION

DEFINITION II.

QUantity may be defined in General, whatever is capable of any ſort of Eſtimate or Menſuration, as immediately the Habitudes and Qualities of Things, as *e. g.* the multitude of Stars in the Heaven, or of Souldiers in an Army, the length of a Rope, or Way, the weight of a Stone, the ſwiftneſs or ſlowneſs of Motion, the Price of Commodities, &c. but mediately, the very things themſelves wherein thoſe Eſtimable Qualities are inherent. Whence with the ingenious *Weigelius* we may not incongruouſly reduce them all to theſe four Kinds or Genders, *viz.* 1. to *Quanta Naturalia*, Natural Quantities, or ſuch as Nature has furniſh'd us with, as Matter with its Extenſion and Parts, the Powers and Forces of Natural Bodies, as Gravity, Motion, Place, Light, Opacity, Perſpicuity, Heat, Cold, &c. 2. to Moral *Quanta* or Quantities, depending for the moſt part on the Manners of Men, and arbitrarious Determinations of the Will; as for Example, the Values and Price of Things, the Dignity and Power of Perſons, the Good or Evil of Actions, Merits and Demerits, Rewards and Puniſhments, &c. 3. to *Quanta Notionalia*, ariſing from the Notions and Operations of the Underſtanding, as *e.g.* the amplitude or narrowneſs of our Conceptions, univerſality or particularity, &c. in Logick; the length or brevity of Syllables, Accent, Tone, &c. in Grammar: And laſtly, to *Quanta Tranſcendentia*, Tranſcendent Quantities, ſuch as are obvious in Moral, Notional, and Natural Beings; as Duration, *i. e.* the Continuation of the Exiſtence of any Being; which in Phyſicks eſpecially is named *Time*, and may be conceived as a Line, &c. To theſe you may moreover add *Unity*, *Multitude* or *Number*, *Neceſſity* and *Contingency.*

DEFINITION III.

NUmber (whereon we ſhall make ſome ſpecial Remarks) if it be taken in the Concrete, is nothing elſe than an Aggregate or Multitude of any ſort of Beings; taken abſtractedly, it is, as *Euclid* calls it, *μονάδων ποσότης*, a *multitude*, or (as they call it) *Quotity of Unities*, on the one hand Number, *i. e.* *many* are oppoſed to *one*; and in that ſenſe Unity is not a Number:

ber: On the other hand Unity may be eſteem'd a Number, ſince it is no leſs (if I may be allow'd that term) ſome *Quotity* than two or three. But as we denote or ſignifie particular Things, when we ſpeak of them Univerſally, by the Letters of the Alphabet, A, B, C, (a, b, c,) *&c.* as univerſal Signs or Symbols of them; ſo for diſtinctly and compendiouſly Expreſſing the innumerable Variety of Numbers, Men have found out various Notes, the moſt natural whereof, are Points diſpoſed in particular extended Orders, as . . . to denote Three, $\begin{matrix}\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{matrix}$ to denote Nine, *&c.* But that way which is moſt commodious for Practiſe, is by the common Notation, or Cyphers, 1, 2, 3, 4, 5, 6, 7, 8, 9. the invention whereof, as we have it by Vulgar Tradition, is owing to the *Arabians*. By a very few of theſe we expreſs any number tho never ſo great, by a wonderful, tho now adays familiar, Artifice; the firſt Inventor of them having Eſtabliſh'd this as an arbitrary Law, that the firſt of them ſhall ſignifie *unity* or *one*; the 2d *two*, &c. as often as they ſtand alone; but placed in a row with others, or on the left hand of one or more 0, or noughts, (which of themſelves ſtand for nothing, but fill up empty places) if in the ſecond, before a nought, they denote Tens; if in the 3d. Hundreds; in the 4th Thouſands; in the 5th Myriads or Tens of Thouſands; in the 7th ſo many Thouſands of Thouſands, or Millions; in the 8th Tens of Millions, *&c.* and ſo onwards, increaſing always in decuple Proportion, by Tens, Hundreds, Thouſands, *&c.*

COROLLARY I.

HEnce you have a way of expreſſing or writing any Sum by theſe Notes, which you may hear expreſſed in Words; as if we were to expreſs in Notes the year of our Lord, One Thouſand Six Hundred Ninety and Nine, it is manifeſt, that according to the method above deſcribed, by placing 9 on the right hand in the firſt place, and nine again in the ſecond towards the left, ſix in the third, and one or unity in the fourth, the buſineſs will be done. Thus it will be eaſie to any one with a little attention, to expreſs any Number whatſoever by theſe Notes; (as ſuppoſe that which *Swenterus* propoſes, in *Delic.*

Phyſico-Math. Part. 1. Probl. 75.) Eleven Thouſand, Eleven Hundred, and Eleven.

COROLLARY II.

HEnce you have alſo the Foundation and Reaſon of the Rule of *Numeration* in Arithmetick, or expreſſing any Numbers or Sum in Words, which you ſee written in Cyphers; which for greater eaſe may be done thus, *viz.* beginning from the firſt Figure towards the right hand, over every fourth Figure note a Point, (including always that which was laſt pointed) and at every ſecond Punctation or Point, draw ſhort ſtrokes thus ′, one over the 2d, two over the fourth, &c. the firſt denoting Millions, the ſecond Millions of Millions, or Bimillions, the third Trimillions, &c. and the Intercepted Points the Thouſands, in their kinds, &c.

SCHOLION.

HEre I cannot omit, on this occaſion, what the forementioned *Weigelius* has hinted about another way of *Numeration*, and which Dr. *Wallis* mentions, *Oper. Mathemat.* Part 1. p. 25. & 66. ſhewing there a way (and illuſtrating it by Examples) of *Numeration*, and of Expreſſing the Figures; which proceeds thus; whereas now adays in numbring we aſcend from unity or 1 to ten (the reaſon whereof; after which *Ariſtotle* makes a prolix Inquiry, *Probl.* 3. *Sect.* 15. was taken without doubt from the denary Number of our Fingers) if from unity we proceed only to four, (which *Ariſtotle* in the ſame place tells us ſome of the *Thracians* uſed to do of old,) and thence returning back again to Unity, we ſhould proceed again after the ſame way; we might after that way obtain a vaſtly more ſimple and eaſie Arithmetick, than we have now adays. Which, even hence we may conclude; becauſe for Multiplication and Diviſion there would need no other Table (or *Pythagorick Abacus*) than this eaſie and ſhort one:

1 . 1 . 1 once one is one;
2 . 2 . 10 *i. e.* twice two are four;
2 . 3 . 12 twice three are four and two.
3 . 3 . 21 thrice three are twice four and one

And altho it is pity that we can't hope now a-days to ſubſtitute this vaſtly eaſier way of Computing, in room of the other now in uſe, becauſe the other is univerſally receiv'd, and moſt ſorts of Meaſures and other Quantities are fitted and accommodated to the decuple Proportion; yet it ought not to be altogether neglected in Mathematicks, which might receive very great advantage hereby, eſpecially in Trigonometry, if the ingenious Invention of the Logarithms had not already ſupply'd its uſe therein. The whole Foundation of this *Tetractys* or *Quaternary Arithmetick*, is placed only in theſe three Notes, 1, 2, 3; ſo that any one of them alone, or in the firſt place, ſhould denote Units, in the ſecond place, Tetrads, or ſo many Fours (or Quaternions,) in the third place ſo many Sixteens, in the fourth place ſo many times Four Sixteens, or 64 s. *&c.* always proceeding in a Quadruple Proportion. For which way of Numeration there might be found out terms as commodious as thoſe we now uſe, and which are thereby grown Familiar to us, as one, ten, twenty, a hundred, a thouſand, *&c.* which will be evident by what follows:

One,	*Unum*	1	One.
Ten,	*Decem*	10	*Quatuor, Tetras, a Quaternion or* (Four,
Twenty,	*Viginti*	20.....	a *Biquaternion*
Thirty,	*Triginta*	30 ...	*a Triquaternion*
Hundred,	*Centum*	100 ...	*a Tetraquaternion*
Thouſand,	*Mille*	1000 .:	*a Quartan*
Ten Thouſand,		10000 *&c.*	*a Tetraquartan*, &c.

DEFINITION IV.

A *Magnitude* is whatever is conceived to be Extended or Continuous, or has parts one without another, and contained within ſome common Term or Terms: wherein that is called a *Point* which is conceived (as indiviſible, or) to have no Parts, and ſo no Magnitude, but is notwithſtanding the beginning or firſt Principle of all Magnitude.

DEFINITION V.

IF we conceive a Point (A) (*Fig.* 1.) to be moved towards B, by this motion it will leave a trace, or deſcribe the Magnitude AB of one only Dimenſion, that is Length without Latitude, or which at leaſt we are to conceive ſo, and is called a *Line*: If that Line AB be conceiv'd again ſo to be moved, as that its extreme Points AB ſhall deſcribe other Lines BC and AD, it will deſcribe by that Motion the Magnitude AB CD, or (to denote it more compendiouſly by the Diagonal Letters) AC or BD, having both length and breadth, but without any depth or thickneſs, or at leaſt ſo to be conceived, and this is called *Superficies* or *Surface*: Laſtly, if this Surface AC be conceived ſo to move, *e. g.* upwards or downwards, that its oppoſite Points A and C again deſcribe other Lines AF and CH, and conſequently each of its Lines other Surfaces, *&c.* by this Motion there will be formed a Magnitude of three Dimenſions, which we call a *Solid* or *Body*, which we will alſo denote by the two diametrically oppoſite Letters AH and DG. But as this Motion of the Point, Line, or Surface, may be various, ſo there will be produced by them various ſorts of Lines, Surfaces, and Solids: But theſe Productions ſtop here, and proceed no further; for the Motion of a Body can only produce another Body greater than the firſt, but no more new Dimenſions.

CONSECTARYS.

I. POints therefore being moved thro' equal Intervals in the ſame or a like way or trace (*e. g.* in a ſtreight or the ſhorteſt trace) deſcribe equal Lines; and

II. The ſame or equal Lines moved thro' the ſame Right-lined or Curvilinear Paths, deſcribe equal Surfaces; and

III. Equal Surfaces moved according to the ſame Methods and Conditions deſcribe equal Solids: which, if rightly underſtood, are the firſt certain and infallible Foundations of the Method of Indiviſibles. But here you muſt take care to diſtinguiſh between the way which the Line it ſelf deſcribes, and that which its Ends or extreme Points deſcribe: For altho *e. g.* the Point *a* (*Fig.* 24.) moves along in a more oblique way than A, and ſo deſcribes a longer Line *a c*; yet the Line *a b* deſcribes by a parallel Motion, an equal ſpace with the Line AB, (*viz.*) the ſame which the whole Line A b, whereof they are parts, would deſcribe. See *Faber*'s *Synopſis*, p. m. 13.

DEFINITION VI.

BUT that we may a little further proſecute this Geneſis of Magnitudes (as very much conducing to underſtand their Nature and Properties) if the Point A moves to B the ſhorteſt way, it deſcribes the Right Line AB; but if in any (one) other it will deſcribe the Curve or Compounded Line ACB: From whence, with *F. Morgues*, we may infer theſe

Fig 2

CONSECTARYS.

I. THat two Right Lines (*a*) beginning from the ſame Point A, and ending in the ſame Point B, will neceſſarily coincide, nor can they comprehend or incloſe Space; for if they did, one muſt deviate, and ſo would ceaſe to be a Right Line.

(*a*) *Eucl. Ax.* 14.

II. In a Space comprehended by three Right Lines AB, BC, CA, (*a*) any two taken together, muſt needs be greater than any one alone. Moreover we may add this before hand;

III. In a Circle a Right Line drawn from A to B (*Fig.* 3.) will fall within the Circle, becauſe the Curve Line ADB deſcribed, as we ſhall hereafter ſhew, being longer than a Right Line, muſt neceſſarily fall beyond it, or on the outſide of it. And laſtly,

IV. A Tangent, or Line (*b*) which does not cut or enter into the Circle, touches it only in one Point.

Moreover if a Right Line AB (*Fig.* 4.) move on another Right Line BC, remaining in the ſame Poſition to it, it will generate a *Plan Surface*, to which a Right Line being any way applied, will touch it with all its point, as *Faber* rightly deſcribes it; if a Right Line be moved on a Curve, or a Curve on a Right Line, *&c.* they will generate a *Curve Surface*, call'd *Gibbous*, or *Convex* without, and *Concave* within.

DEFINITION VII.

IF a Right Line be fixed at one of its ends A, and the other be moved round (*Fig.* 5.) it will deſcribe in this Motion

(*a*) *Eucl. l.* 1. *Prop.* 20.
(*b*) *Eucl. Prop.* 2. *lib.* 3.

Circular Plane, or a Circle; and by the motion of its end or extreme Point B, the *Periphery* or *Circumference* of that Circle BEF. The fixed Point A is called the Center of that Circle; the Lines AB, AC, *&c.* its *Radii* or *Semi-Diameters*; all of which are equal one to another. Any Right Line BC drawn from one part of the Circumference thro' the Center to another, is called the *Diameter*, and divides the Circle into two Semicircles BECB and BFCB. The Circumference of a Circle, whether great or ſmall, is divided into 360 equal parts called Degrees, and each Degree into 60 Minutes, *&c.* From this Geniture of the Circle preſuppoſed, there evidently follow theſe

CONSECTARYS.

I. THat 2 Circles which cut one another cannot have the ſame common Center; for if they had, the *Radii* ED and EA drawn from the common Center E (*Fig.* 6.) would be equal to the common *Radius* EB that is the part to the whole.

II. Nor can two Circles touching one another within ſide, have one and the ſame Center, for the ſame reaſon.

(*a*) *Eucl.* 5 *l.* 3
(*b*) & 6 *of the ſame Book.*
(*c*) *Eucl.* 8 *l.* 3.

III. Of Lines falling from any given Point without the (*Fig.* 7.) Circle, and (*c*) paſſing thro' the Periphery to the oppoſite Concave part of it, that which paſſes through the Center of it, is the longeſt, *viz.* AB; and of the other that which is neareſt to it is longer than that which is more remote: But on the contrary of thoſe which fall on the Convex Periphery, that which tends towards the Center, as Ab, is the leaſt, and the reſt gradually greater, and there can be but two, as AE and AF, or Ae and Af, equal: All which will appear very evident by drawing other Circles from the Center A thro' B,D,E, and *b, d, e.* Or thus; having drawn two other Circles, from the Radii AB and A b, if we conceive the Radii Ab and Cb to move towards the right hand, their ends will always recede further from one another; the ſame is alſo evident of the Radii AB and CB, moved alſo to the right together.

IV. Moreover (*Fig.* 8.) of all the Lines drawn within the Circle (*a*) the Diameter is the greateſt, and the reſt gradually leſs, by how much the more remote they are from the Center, *&c.*

Which will be very evident to any one who contemplates a Circle inſcribed in a Square, as alſo the Geneſis of Curvity it ſelf; as alſo many other ways which I ſhall now omit; or to mention one more thus; becauſe the two Radii CA and CB being moved, in order to meet together, neceſſarily approach nearer to one another in their extreme Points.

DEFINITION VIII.

THE Aperture or opening of two Lines (*Fig.* 9.) AB, AD, *&c.* that are both fixed at one end at A, and the other ends opened or removed farther and farther from one another, is called an *Angle*, and uſually denoted by 3 Letters, D, A, B, (whereof that which denotes the Angular Point, always ſtands in the middle,) and meaſured by the Arch of a Circle BD, or a certain number of Degrees which it intercepts. The greateſt Aperture of all BAC is when the 2 Legs of the Angle AB and AC make one Right Line, and is meaſured by a Semicircle, or 180 Degrees. The mean or middle Aperture BAE or CAE, when one Leg EA is erected on the other AB or AC at Right Angles, ſo that it inclines neither one way nor the other, (thence called a *Perpendicular*) is named a *Right Angle*, whoſe meaſure is conſequently a Quadrant (or quarter part) of a Circle or 90 Gr. Wherefore a Semicircle is the meaſure of two Right Angles: An Aperture or Angle BAD leſs than a Right Angle (and ſo meaſured by leſs than 90 degrees) is called an *Acute Angle*; and that which is greater than a Right Angle, as DAC (and ſo conſiſting of more than 90 degrees) is called an *Obtuſe Angle.* Whence me may now draw theſe

(a) *Prop.* 15. *lib.* 3.

COROLLARYS.

I. TWO or more Contiguous Angles (*a*) conſtituted on the ſame Right Line BC, and at the ſame Point A (as DAB, and DAC or DAB, DAE and EAC) make two Right Angles, as filling the Semicircle; and conſequently,

(a) *Eucl.* 13. *lib.* 1. *with the Coroll.*

II. All the Angles that can be conſtituted about the Point A (as filling the whole Circle) are equal to 4 Right ones: As alſo on the other ſide *(a)* if two Right Lines AB and AD meet on the ſame point A of another Right Line A C, and make the Contiguous Angles equal to 2 Right ones, that is, if they fill a Semicircle, BC will neceſſarily be the Diameter of a Circle, and conſequently a Right line.

III. If one of the Contiguous Angles BAE be a Right one, the other CAE will be ſo alſo.

IV. If two Right Lines AB, CD, cut one another in E, the 4 Angles they make will be equal to 4 Right ones.

V. And as it is evident at firſt ſight (*Fig.* 10.) that any Circle having one half (or Semicircle) folded on the other, at the Diameter ECD, the two Semicircles EHD, and EID, muſt needs agree, or every where coincide one with the other; ſo if the Angle ACD be ſuppoſed equal to the Angle BCD, that is, the Arch AD to the Arch BD, having one Leg CK or CL common; the others AC and BC being ſuppoſed before equal,

1. The Baſes BL and AL, KB and KA, will be alſo equal; for theſe will coincide too, and therefore the Angles alſo.

2. The Line A B being biſected in K, the two Angles (*c*) at K will alſo coincide and be equal, and conſequently Right Angles: and contrarywiſe,

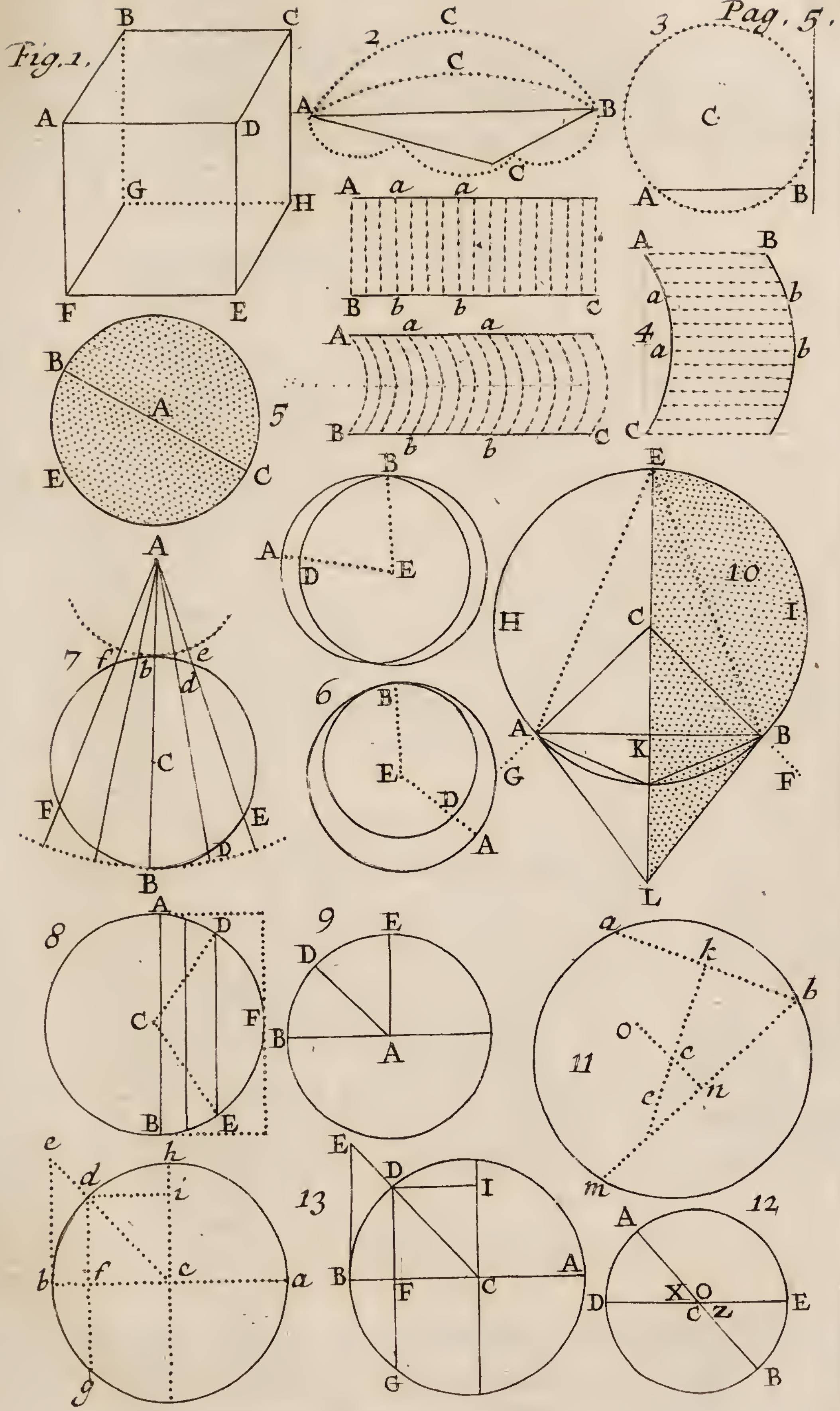
3. The Angles at the Baſe of equal (*d*) Legs, CAB, CBA, and alſo thoſe below the Legs, the Legs being produced to F and G, are equal.

4. Conſequently the Spaces ACL and BCL, ACK and BCK are equal to one another.

(*a*) *Euclid. l.* 1. *Prop.* 14.
(*b*) *Eucl. l.* 1. 4 & 3.
(*c*) *lib.* III. 3.
(*d*) *lib.* 1. 5.

5. The Contiguous Angles AED and BED inſiſting on equal Arches AD and BD are equal, and *è contra*; as alſo thoſe that are not Contiguous, if their Vertex's are equidiſtant from E, *&c.*

6. It is hence alſo manifeſt, that a Perpendicular erected on the middle of any Line AB, inſcribed in any Circle, paſſes through its Center, by what we have juſt now ſaid; and if you likewiſe erect Perpendiculars on the middle of any 2 Lines, *ab* and *bm* (*Fig.* 11.) connecting any 2 Arches, or any 3 Points, *a*, *b*, *m*, that are not placed all in the ſame Right Line, thoſe 2 Perpendiculars *ke*, *no*, will determine (by their Interſection) the Center of a Circle that ſhall paſs through theſe 3 Points.


Pag. 5.
Fig. 1.
A
B
C
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3
C
A
B
A a a
B b b C
A a a
B b b C
4
A B
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a b
C
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DEFINITION IX.

IF one Right Line DE cut or paſs thro' another AB (*Fig.* 12) the oppoſite Angles at the top or interſection ACD and ECB are called *Vertical*; as alſo the other two ACE and DCB: Whence follow theſe

COROLLARYS.

I. THat the Vertical Angles are always (*a*) Equal; for both ACD and ECB with the third, ACE, which is common to both, fill or are equal to a Semicircle; as likewiſe both ACE and DCB with the third ECB, which is common.

(*a*) *Eucl. lib.* 1. 15.

II. Contrarywiſe, if at (*a*) the Point C of the Right Line DE, the 2 oppoſite Lines AC and CB make the Vertical Angles *x* and *z* equal, then will AC and CB make one Right Line; for, ſince *x* and *o* make a Semicircle, and *z* and *x* are equal, by Hypoth. *o* and *z* will alſo make or fill a Semicircle, whoſe Diameter will be ACB.

III. By the ſame Argument it will appear, that of 4 Lines (*b*) proceeding from the ſame Point ſo as to make the oppoſite Vertical Angles equal, the 2 oppoſite ones AC and CB, as alſo DC and CE, will make each but one Right Line; for ſince all the 4 Angles together make a whole Circle, or 4 Right Angles, and the ſum of *x* and *o* is equal (by Hypoth.) to the ſum of *o* and *z*, it follows, that both the one and the other will make Semicircles, whoſe Diameter will be AB and DE, and ſo Right Lines.

(*a*) *The ſame Prop.* Schol. 1.
(*b*) *Schol.* 2.

DEFINITION X.

IN any Circle, a Right Line, as DG, that ſubtends any Arch of it DGB, is called the *Chord* of that Arch (*Fig.* 13.) BF (a part cut off from the Semidiameter BC paſſing thro' the middle of the Chord) is called the *Sagitta* or *Intercepted Ax*, but moſt commonly the *Verſed Sine*; and DF let fall from the other extremity of the given Arch BD, on the Semidiameter at Right Angles, is called the *Right Sine* of that Arch BD, or of the Angle

BCD; alſo DI is called the Right Sine of the Complement (or for brevity ſake, Sine Compl.) of that Arch DH, or Angle DCH, *&c.* but the greateſt of all the Right Sines HC let fall from the other extremity (or end) of the Quadrant (which is indeed the ſame as the Semidiameter of the Circle) is called the *whole Sine* or *Radius*; laſtly, BE is called the *Tangent* of the Arch BD or Angle BCD, and CE its *Secant:* Whence Mathematicians, for the ſake of Trigonometrical Calculations, have divided the whole Sine or Radius of the Circle into 1000, 10000, 100000, 1000 000, 10000 0000, *&c.* parts, thence to make a proportionable Eſtimate of the number of Parts in the Sine, Tangent, or Secant of any Arch, *&c.* as may be ſeen in the Tables of Sines, Tangents, and Secants. From theſe Suppoſitions and Explications of the Terms, we ſhall now infer from this Definition the following

COROLLARYS.

I. IN equal Circles (and ſo much more (*a*) in one and the ſame) as the *Radii* or Semidiameters BC and *b c* are equal, ſo alſo it is evident, that the *Right Sines* DF & *Df*, of equal Arches BD and *b d*, or equal Angles BCD and *bcd*, alſo the Tangents BE and *b e*, and Secants CE and *c e*, and Subtenſes or Chords DG and *d g*, alſo the Sagittæ or intercepted Axes BF and *b f*, of double the Arches DBG and *d b g*, &c. will be equal, and ſo conſiſt of an equal number of Parts of the whole Sine or Radius, *&c.* which both is evident from what we have ſaid before, and may be further evinced, if one Circle be conceived to be put on the other, and the Radius BC on the Radius *b c*, that ſo they may coincide, by reaſon of the equality of the Arches BD and *b d*; and ſo of all the reſt. And *è contra*,

(*a*) *Among the reſt* Vid. *Eucl.* 28 & 29 *lib.* 3, *and alſo partly the* 26 *and* 27.

II. In unequal Circles, the Sines, Tangents, *&c.* of equal Angles BCD or *b c d* (*Fig.* 14.) or ſimilar Arches, or Arches of an equal number of Degrees, BD and *b d*, will be alſo ſimilar or like, *&c. i. e.* the Sine *df* conſiſts of as many parts of its *Radius bC*, as the Sine DF does of its Radius BC, *&c. e. g.* if the Radius BC be double of the Radius *b c*, each thouſandth part of the one, will be double of each thouſandth part of the other, but they are alike 1000 in each; becauſe the degrees in the Circumference of the little one, particularly in the Arch *b d* are but half as big as thoſe in the Arch BD, and yet equal in number in both. Thus alſo if the Sine DF contains 700 of the 1000 parts of its Radius BC, *df* will alſo contain 700 of the 1000 parts of its ſmaller Radius *bc*, and in like manner the Chords DG and *dg*, and the Tangents BE and *be*, *&c.* contain a like number of parts, each of its own Radius.

SCHOLION.

IT may not be amiſs here to note by the by (altho it may ſeem more proper to be taught after the Doctrin of Proportions) that if, *v. g.* the degrees of a greater Circle be each of them reſpectively double, or triple, or quadruple, *&c.* of the degrees of a leſs Circle, according as the Radius of the one is double or triple to the Radius of the other, then, at leaſt as far as Mechanical Practice can require, you may find the Arch of a greater Circle equal to the whole Periphery of a leſs, *viz.* if you take reciprocally that part of the greater Periphery, which ſhall be as the Radius of the leſs to the Radius of the greater, or as one degree of the leſs Periphery to one degree of the greater. *e.g.* if the leſs Radius *bc* be half the greater BC, and ſo alſo the Periphery, and each of the degrees of the one, be one half of the Periphery, and of each of the degrees of the other, one half of the greater Periphery will reciprocally be equal to the whole leſs Periphery, or 180 degrees of the one to 360 of the other, *&c.*

2. The ſame (at leaſt in this caſe where the Radius *c b* is double of the Radius CB) may be done alſo Geometrically by the ſame reaſon. Having deſcribed Circles on each Radius, ſuppoſe the Radius CB (*Fig.* 15.) ſo to move with an equable motion about its Center *c*, as to take or move the Radius of the greater Circle *c b* along with it, and coming, *v.g.* to I. ſtops that alſo at 1, and going forward, to II. ſtops that again at 2, *&c.* Hence it will be manifeſt to any attentive Reader, that when the leſs Radius CB ſhall have deſcribed the Semicircumference B. II. III. the greater Radius *cb* having moved to 3, will have deſcribed preciſely a quarter of its circumference;

and if ſtill the leſs Radius C. IV. moves on to the right hand, and continues to carry the greater *c*4 along with it onwards the ſame way, it will neceſſarily follow, that in the ſame moment as the Radius C. IV. (together with *c* 4.) ſhall come to its firſt ſituation in B, having deſcribed a whole Circle, the oppoſite Radius *c*4 will be come to 5, and have deſcribed half its Circle, having moved all along with an equal Motion. Hence it is evident, that the whole leaſt Circle anſwers exactly to half the greater, and half of the firſt to a quarter of the laſt; as alſo the Quadrant B. II. to the Octant (or 8th. part) *b* 2, &c. whence any Arch being given, as B. I. in the leaſt Circle, if you draw thro' I the Radius of the greater Circle *c* 1. you'l cut off an Arch *b* 1. equal to the given Arch in magnitude, but only half in the number of degrees.

3. Hence follows naturally that celebrated Propoſition of *Euclid*, that the Angle at the Center BC. I. or BC. II. is double of the correſponding Angle at the Periphery *bc.* 1. or *bc.* 2, *&c.* which in this caſe is manifeſt, and in the other 2 (*Fig.* 16.) of the wholes or remainders DCD and DPD it is alſo (*a*) certain; which is true alſo of the parts *B*CD and *B*PD to be added or ſubtracted by the firſt Caſe.

4. Hence we have a new way of biſecting any given Angle CDE, or Arch CE (*Fig.* 17.) *viz.* if you make C*B* equal to the Leg DC, and from this, as Radius, deſcribe an Arch *B* F equal to the Arch CE, and draw DF.

And with the ſame facility we might obtain the Triſection, if the greater Radius being triple to the leſs, was thus carried along by an equable Motion, as we have ſhewn how to do already in a double Radius; and this at firſt ſight may ſeem very probable.

*B*ut whether the triple Radius be immediately carried round by the ſimple Radius C*B*, or by means of the double Radius *c b*, neither the one nor the other will cauſe an equable Motion. For in the latter Caſe, while the Radius *c b* deſcribes the quadrant *Bb*, the Radius *d e* will not deſcribe ſo much as a Quadrant; but while *c b* with the ſame velocity deſcribes the other Quadrant *b f*, the Radius *d e* will come to *g*, deſcribing an Arch as much greater than a Quadrant as the for-

(*a*) *Eucl. p.* 20. *l.* 3.
(*b*) *Eucl. p.* 9. *l.* 1.

mer was leſs. In the former caſe on the contrary, the Radius CB moved on to D beyond a Quadrant, while *de* was carried from *B* to *e*, but if the Radius CD moving on, ſhould again carry *de* along with it, the one would deſcribe the ſame Arch *eB*, while the other would deſcribe one leſs than before.

5. Hence the Angle at the Center ACE (*Fig.* 19.) upon the Arch AE, is equal to the Angle *ADB* at the Periphery, upon double that Arch *AB*.

6. Hence the Angle *ADB* in the Semicircle (Num. 1.) (*a*) is a Right Angle, in a Segment leſs than a Semicircle (Num. 2.) is an obtuſe Angle, and in a greater (Num. 3) an Acute one, becauſe the Angle at the Center *ACE* upon the half Arch, is equal to the Angle *ADB pr. præced.* 5. and is a Right Angle in the firſt Caſe, Obtuſe in the ſecond, and Acute in the third.

(*a*) *Eucl.* 31. *lib.* 2.

7. Hence Angles in the ſame Segment, or (*b*) on equal Segments of equal Circles, or on the ſame or equal Arches, are all equal and *è contra.*

(*b*) *Eucl.* 26 *and* 27. *lib.* 3.

DEFINITION XI.

WHen 2 or more Lines *AB* and *CD* are ſo continued as to keep always the ſame diſtance from one another (whoſe Geneſis may be conceived to proceed from the uniform Motion of 2 Points *A* and *C*, always keeping the ſame diſtance from each other) they are called *Parallels*: But as it evidently follows from this Definition, that (*a*) thoſe Lines which are Parallel to one third, are parallel to one another (ſince adding or ſubtracting equal Intervals to or from other equal ones, the ſums or remainders muſt needs be equal ;) ſo if the Parallels are Right Lines and cut tranſverſly (or ſlopingly a-croſs) by another Right Line *EF*, you'l have theſe

(*a*) *Eucl. lib.* 1. 30.

COROLLARYS.

I. THE Angles (*b*) which we call *Alternate* ones, GHK and HGI (*Fig.* 21.) are equal by *Corollary* I. *Definition* X. ſince the diſtances GK and HI, which are the Right Sines of the ſaid Angles, are ſuppoſed equal.

(*b*) *Eucl. lib.* 1. 29.

II. The External Angle EGA is alſo equal to the Internal oppoſite Angle GHK, by Conſect. 1. *Definit.* 9. becauſe that External Angle EGA is equal to the alternate Vertical one HGI.

III. The ſame Internal Angle GHK, with the other internal oppoſite one on the ſame ſide AGH (as well as the External one EGA, by *Coroll.* 1. *Definit.* 8.) are equal to two Right ones.

IV. On the contrary, If any Right Line EF (*a*) cutting 2 others AB and CD tranſverſly, makes the alternate Angles GHK and HGI equal, their Right Sines, by *Conſect.* 1. *Definit.* 10. will be equal, and conſequently the Lines AB and CD parallel: and the ſame will follow, if the External Angle be ſuppoſed equal to the Internal, or the 2 Internal ones on the ſame ſide equal to 2 Right ones; ſince from either Hypotheſis the former will immediately follow.

(a) lib. 1. Prop. 27 & 28

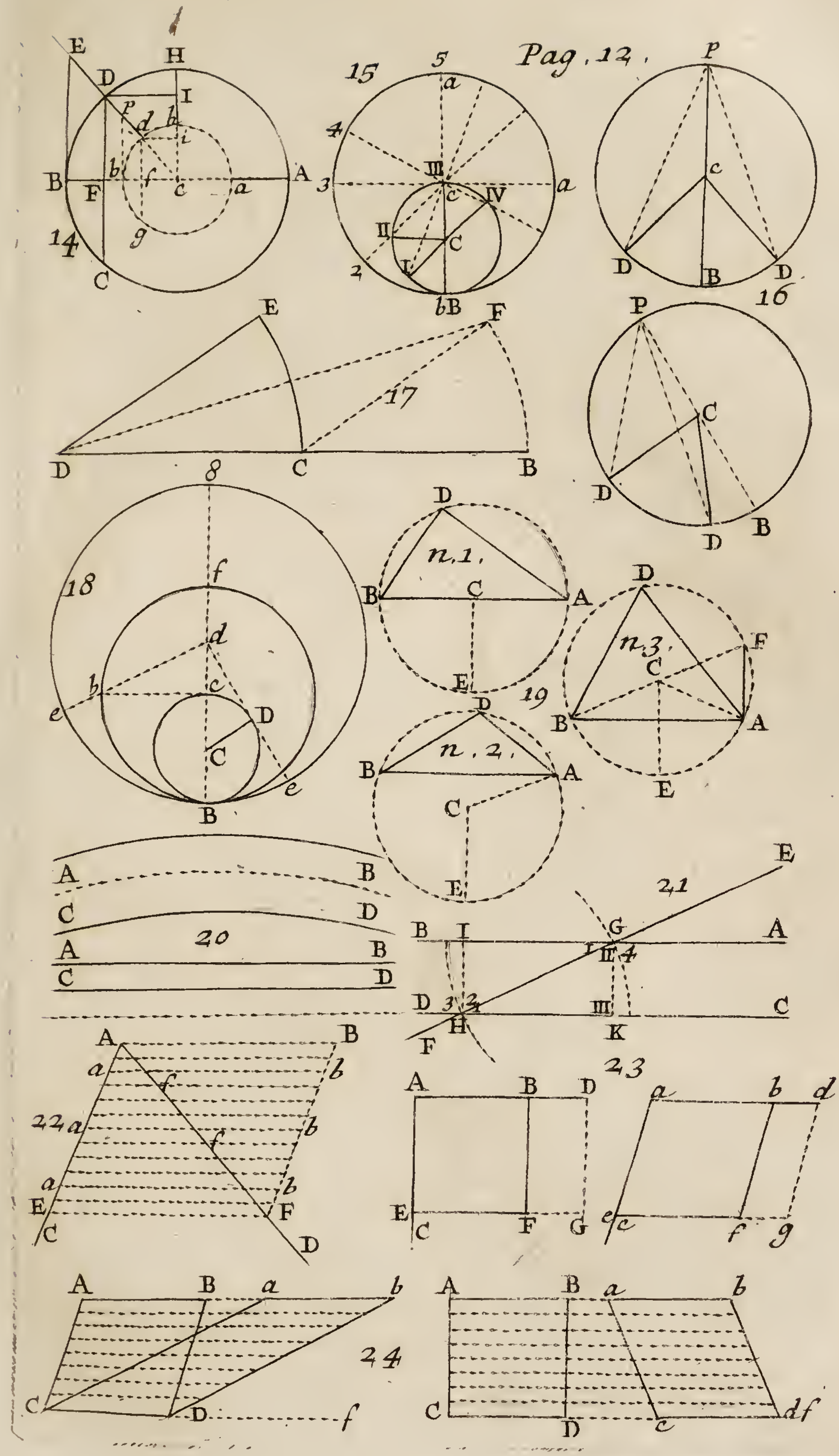
V. From whence it appears more than one way (*b*) *That the 3 Internal Angles of any Triangle* (e. g. H,G,K, which will ſerve for all) *taken together, are equal to two Right ones, and the External one* GHD *is equal to the two Internal oppoſite ones.* For we might either conclude with *Euclid*, that 1, 2, 3, together make 2 Right ones, by *Conſect.* 1. *Definit.* 8. but 2=II and 3=III pr. 1 and 2 of this, therefore I, II, III = 2 Right ones; or with others, 1, II, 4 are =2 R. but 1=I and 4=III pr. 1ſt of this. Therefore, &c. or more briefly with *F. Pardies,* 1=I pr. 1ſt of this, but 1, II, III, together = to 2 Right ones, by the 3d of this; therefore I, II, III = 2R. Q. E. D.

(a) lib. 1. Prop. 32,

DEFINITION XII.

IF a Right Line AB (*Fig.* 22.) be conceived to move from the top of a plain Angle CAD with a motion always parallel to its ſelf, ſo that at one end A it ſhall always touch the Leg AC, and all along cut the Leg AD, while at length being come to F, it ſhall only touch that Leg with its other end B, and ſo fall at length wholly within the Angle CAD: It will deſcribe by this motion within the Legs CAD the Triangular Figure EAF, and without them the Triangular Figure BAF; its parts within them *af* continually increaſing, and the other without

Pag. 12.



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fb continually decreaſing; but with all its Parts, or the whole Line, it will deſcribe the Quadrangular Figure AEFB: Conſequently if the other Leg AD of the given Angle CAD (*Fig.* 23.) or any part of it AB, be moved along the other Leg remaining parallel to it ſelf, it will alſo deſcribe a Quadrilateral Figure, which will be alſo equilateral, if the Line deſcribing it AB, be equal to the Line AE according to which it is directed; but if either of the Lines, as AD be greater than the other, the oppoſite Sides will be only equal; for the deſcribent or deſcribing Line is always neceſſarily equal to its ſelf, and the Points A, B, D, moved with an equal Motion, deſcribe alſo in the ſame time equal Lines AE, BF, DG. From theſe *Geneſes* of Quadrangles and Triangles we have the following

CONSECTARIES.

I. These Quadrilateral Figures are alſo Parallelograms, *i. e.* they have their oppoſite Sides Parallel; (*a*) becauſe the Line that deſcribes them is ſuppoſed to remain always parallel to its ſelf, and the Points A and D, or A and B, to be always equidiſtant.

(*a*) *Schol. Prop.* 34. *lib.* 1.

II. Becauſe the 2 Internal oppoſite Angles (*b*) A and E, and alſo E and F, &c. are equal to 2 Right ones, by *Conſect.* 3. *Definit.* 11. if one Angle *v. g.* that at A be a Right one, all the others muſt neceſſarily be ſo too [in which caſe the quadrilateral and equilateral Figure AF is called a *Square*, and the other AG an *Oblong*, or *Rectangle*:] if there be no Right Angle, the oppoſite Angles tranſverſly or croſs-ways, *a* and *f*, or *a* and *g* are equal, becauſe both the one and the other, with the third (*e*) make 2 Right ones [in which caſe the quadrilateral Equilateral *af* is called a *Rhombus*, but the other *ag* a *Rhomboid*.

(*b*) *The firſt part of the ſame Propoſit.*

III. The Tranſverſal (or Diagonal) Line (*c*) in any Parallelogram, divides it into two equal Triangles AEF and FAB; for all the Lines and Angles on each ſide are equal, and as the deſcribent (Line) AB moved thro' the Angle EAF upon the Line AE deſcribed the Triangle AEF; ſo the Line EF, equal to the former, moved after the ſame way, thro' the Angle AFB alſo equal to the former Angle, upon the equal Line FB, muſt neceſſarily

(*c*) *the latter part of the ſame Prop.*

neceſſarily deſcribe an equal Triangle; or, in ſhort, all the Indiviſibles *a f*, or their whole increaſing Series, are neceſſarily equal to the like number of Indiviſibles *f b*, increaſing reciprocally after the ſame way.

IV. All Parallelograms that are between the ſame Parallels AB and CF (*Fig.* 24.) *i. e.* having the ſame Altitude (*a*) and the ſame or equal Baſes, as CD or CD and *c d*, are equal among themſelves; for they may be conceived to be deſcribed by the equal Lines AB and *a b* equally moved thro' the ſame or equal Intervals of the Parallel Lines; ſo that all or each of the Indiviſibles or Elements AB will neceſſarily be equal to all and each of the Indiviſibles *a b*; for they all along anſwer one to the other both in number and magnitude.

(*a*) *lib.* 1. *Prop.* 35 & 36

SCHOLIUM.

HEre you have a Specimen of the Method of Indiviſibles, introduced firſt by *Bonaventura Cavallerius*, and ſince much facilitated; and altho theſe Indiviſibles placed one by another, or as it were laid upon an heap, cannot compoſe any Magnitude, yet by an imaginary Motion they may meaſure it, and as it were, after a negative way, demonſtrate the Equality of two Magnitudes compared together, *viz.* if we conceive a certain number of ſuch Elements in any given Magnitude, and thence conclude that in another conſiſting of the like Elements, ordered or ranked after the ſame way, there can be neither more nor leſs in number than in the firſt; thence follows their Equality, &c.

V. Hence therefore it is alſo Evident, that Triangles upon the ſame and equal Baſes as CD and *cd*, and placed between the ſame Parallels, are neceſſarily equal, becauſe they are the half of equal Parallelograms AD and *a d*, by the 3d *Conſectary* of this *Definition.*

VI. *F. Mourgues* ingeniouſly concludes from hence, *viz.* becauſe the 2 Internal oppoſite Angles (*b*) on the ſame ſide in any Parallelogram, are equal to two Right ones, and ſo all together equal to four; that therefore the three Angles of any Triangle ABC (*Fig.* 25. which may always be compleated into a Parallelogram)

(*a*) *lib.* 1. *Prop.* 37 & 38, & 41.
(*b*) *lib.* 1. *Prop.* 32. *otherwiſe.*

Parallelogram) are equal to half of thoſe four, or to two Right ones. This may be yet more briefly conceived thus; the Angles $b + c + d$ (the ſum of the two inferiour ones) = to two Right ones; but $a =$ alternate d: therefore $b + c + a =$ to two Right ones, Q. E. D.

VII. Becauſe it is manifeſt in Rectangular Parallelograms, if the Altitude AB, and Baſe BC (*Fig.* 26.) meaſured and divided by the ſame common Meaſure, be conceived to be (multiplied or) drawn one croſs the other, that the * Area AC, thereby deſcribed, will be divided into as many little ſquare Meaſures or Area's, as the number of their Sides multiplied together would produce Units; therefore the Area of any other Parallelogram will be after the like manner produced, if the Baſe be multiplied by the Perpendicular heigth, equally as if it were a Rectangle of the ſame Baſe and Altitude.

* *This is manifeſt from its Geneſis; for the five parts of the Line AB by its motion along the part of the Baſe BE, deſcribes 5 little Squares, and moving along the following part, deſcribes five more, &c.*

VIII. Conſequently alſo you may have the Area of any Triangle, by *Conſectary* 3 and 5. if the Baſe be multiplied by half the Perpendicular heigth; or, the whole Baſe being multiplied by the heighth, if you take the half of the product.

DEFINITION XIII.

BUT as there are various Species of Triangles, while firſt with relation to their Sides, one is called *Equilateral*, as ABC (*Fig.* 27.) becauſe all its Sides are equal; another *Equicrural* or *Iſoſceles*, as DEF, becauſe it has two equal Sides DE and EF, while its Baſe DF may be either longer or ſhorter; and a third is called *Scalenum*, as GHI, becauſe it has all its Sides unequal; then again in reſpect to their Angles, one is called *Rectangled*, as *a*, *b*, *c*, becauſe it has one Right Angle at *a*;

Another *Obtuſangled*, as *d*, *e*, *f*, becauſe it has one obtuſe Angle at *d*; a third is called *Acuteangled*, as *g*, *h*, *i*, becauſe all its three Angles are Acute: So each of theſe kinds has its peculiar properties, which we ſhall partly hereafter demonſtrate in their proper places, and partly deduce here as

CONSECTARYS.

I. ALL Equilateral Triangles are alſo Equiangular, and conſequently Acutangled; for, having found a Center for three Points, and the Periphery A, B, C, (ſee *Fig.* 28.) by *Conſect.* 6. *Definit.* 8. the three Arches AB, BC, and AC, anſwering to equal Chords; and conſequently the three Angles at the Center O are equal, by *Conſect.* 1. of *Definit.* 10. and therefore the three Angles at the Periphery alſo, as being half of the other, by the 3d *Conſectary* of the ſame *Definition.* Each Angle therefore is one third part of two Right ones, by *Conſect.* 6. *Definit.* 12. two thirds of one Right Angle, *i. e.* 60 degrees, and conſequently Acute.

II. It follows alſo by the ſame Reaſon in an *Iſoſceles* Triangle, that the Angles at the Baſe oppoſed to equal Sides are equal, and (*a*) conſequently Acute; for having circumſcribed a Circle about it, equal Arches will correſpond to the equal Chords DE and EF, and equal Angles at the Center DOE and FOE will correſpond to them, and equal ones at the Periphery DFE, and FDE to theſe again. And it is evident that each of theſe are leſs than a Right Angle *h. e.* an Acute one, becauſe all three are equal to two Right ones. Wherefore if the third is a Right Angle, the other two at the Baſe will neceſſarily be half Right ones.

(*a*) *lib.* 1. *Prop.* 5. *the ſame otherwiſe demonſtrated.*

SCHOLIUM.

WE will here (*a*) ſhew by way of Anticipation, the truth of the *Pythagorick Theorem*, eſteemed worth an *Hecatomb*: Which hereafter we will demonſtrate after other different ways; *viz. In a Right Angled Triangle BAC* (*Fig.* 29.) *the Square of the greateſt Side oppoſite to the Right Angle, is equal to the Squares of the other two Sides taken together.* For having deſcribed the Squares of the other two Sides, AC *d E*, DE *a b* (taking ED = A*B*) and the Square of the greateſt *B*C *cb*, it will be evident, that the parts X and Z are common to each, and that the two other Triangles in the greateſt Square *BAC* and *BDb*, are equal to the two Triangles *b a c* and C *d c* which remain in the two Squares of the leſſer Sides; and ſo the whole truth of the Propoſition will be evident, while theſe two things are undoubtedly true: 1. That the Side of the greateſt Square *Bb* will neceſſarily concur with the Extremity of the leſs *Db*, and the other Side of the greateſt Square *Cc* with its Extremity *c*, will preciſely touch the Continuation of the Sides of the two leaſt Squares *dEa*; as you'l ſee them both expreſſed in the Figure. 2. The ſaid two Triangles are every way equal; for the Angles at C with the intermediate one at Z, make two Right ones, therefore they are equal; but the Side *CA* is equal to the Side *cd*, and *CB* to *Cc*, and the Angles at *A* and *d* Right ones. Wherefore if we conceive the Triangle *ABC* to to be turned about C, as a Center to the right hand, it will exactly agree with the Triangle *Cdc*, and the Point *B* will neceſſarily fall on the continued Line *dE*, as agreeing with the Line *AB*. Hence it is now evident, that $Ca = BD$, and becauſe *ba* is alſo $= bD$, and the Angles at *a* and *D* Right ones. Where, if we conceive the Triangle *bac* to be moved about *b* as a Center, untill *ba* coincides with *bD*, and *ac* with *DB*, *bc* will alſo neceſſarily coincide with *bB* *Q. E. D.*

To this Demonſtration of *Van Schooten*'s, which we have thus illuſtrated and abbreviated, we will add another of our own, more like *Euclids*, but ſomewhat eaſier, which is this: Having drawn the Lines (as the other Figure 29 directs) the △ *ACD* being on the ſame Baſe *AC* with the Square *AI*, and between the ſame Parallels, is neceſſarily one half of it, but it is alſo half of the Parallelogram *CF* being on the ſame Baſe with it, *viz.* *DC*; therefore this Parallelogram = ▭ *AI*. In like manner △ *ABE* is half the ▭ *AL*, and alſo half the Parallelogram *BF*, therefore *BF* = ▭ *AL*: therefore *CF* + *BF* that is the ▭ of *BD* = to the two ▭▭ *AI* + *AL*. *Q. E. D.* For becauſe the Side *BE* occurs to, or meets the Side *LK*, and the Side *CD* the Side *IH* continued, it yet more apparently follows; becauſe the Angles *a* and *b*, and alſo *c* and *d*, are manifeſtly equal, as making both ways, with the Intermediate *x* or *z*, Right Angles. Therefore the △ *BAC* being turned on the Center *B* and laid on *BLE* will exactly agree with it, and turned on the Center *C* and laid on *CID*, will agree with that alſo, &c.

DEFINITION XIV.

ALL Rectilinear or Right Lined Figures that have more than three or four Sides (to the latter ſort of which, there remains to be added another Species beſides Parallelograms, call'd *Trapeziums*, whoſe Angles and Sides are unequal, as K, L, M, N, *Fig.* 30.) are called by one common Name *Poligons*, or *Many-ſided* and *Many-Angled Figures*, and particularly according to the Number of their Sides and Angles, *Pentagons*, *Hexagons*, *Heptagons*, &c. All whereof, as alſo *Trapezia*, being reſolvible into Triangles by Diagonal Lines, (as may be ſeen in the 31 and foregoing Fig.) you have theſe

CONSECTARYS.

I. YOU have the Area of any Polygon by reſolving it into Triangles, and then adding the Area's of each Triangle found by *Conſect.* of *Definition* 12 into one Sum.

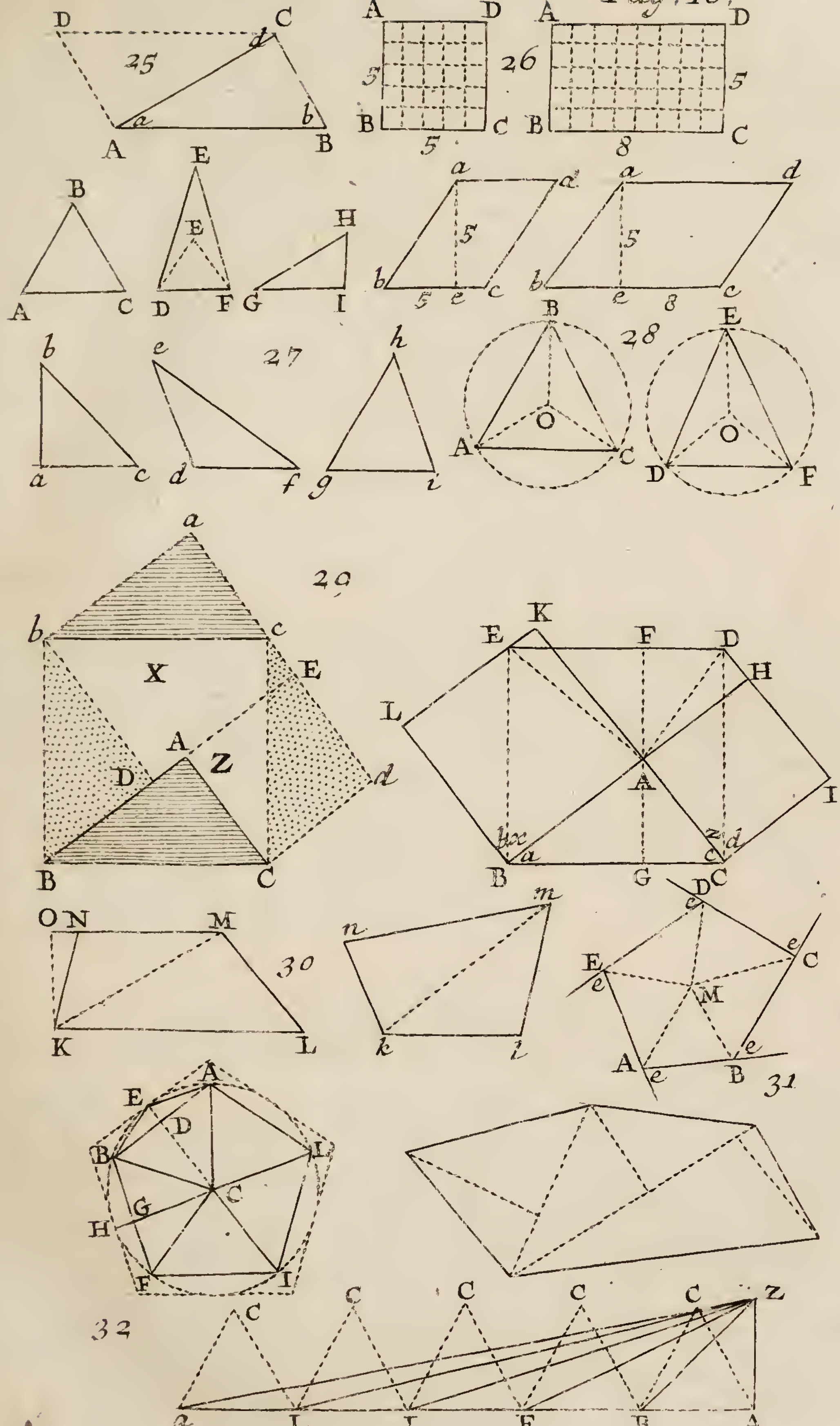
II. The Area of the *Trapezium* KLMN (in the firſt of the *Fig.* 30) whoſe two oppoſite Sides, at leaſt KL and MN, are Parallel, may be had more compendiouſly, if the Sum of the Sides be multiplied by half the common heigth KO.

SCHOLIUM.

HEnce we have the foundation of *Epipedometry* or Maſuring of Figures that ſtand on the ſame Baſe, and Ichnography; in the Practiſe whereof this deſerves to be taken ſpecial Notice of, that to work ſo much the more Compendiouſly, you ought to divide your Figure into Triangles, ſo that (*Fig.* 31.) 2 of their Perpendiculars may (as conveniently can be) fall on one and the ſame Baſe. For thus you'l have but one Baſe to meaſure, and 2 Perpendiculars to find the Area of both: But for Ichnography, the diſtance of the Perpendiculars from the neareſt end of the Baſe muſt be taken; which we ſhall ſuperſede in this Place and Diſcourſe more largely on hereafter.

2. This reſolution of a Polygon into Triangles may be perform'd by aſſuming a point any where about the middle, and making the ſides of the Polygon the Baſes of ſo many Triangles;

Pag. 18.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Triangles; (ſee the 2*d* Figure mark'd 31) wherein it is evident; 1 That all the Angles of any Polygon are equal to twice ſo many right ones, excepting 4, as the Polygon has ſides; for it will be reſolv'd into as many Triangles as it has ſides, and each of theſe has its Angles equal to 2 right ones. Subtracting therefore all the Angles about the Point M (which always make 4 right ones by *Conſ.* 2. *Def.* 8.) there remain the reſt which make the Angles of the Polygon. 2 All the external Angles of any right lined Figure (e, e, e, *&c.*) are always equal to 4 right ones; for any one of them with its Contiguous internal Angle is equal to 2 right ones *pr. Conſect.* 1 of the ſaid *Def.* and ſo altogether equal to twice ſo many right ones as there are Sides or internal Angles of the Figure. But all the internal Ones make alſo twice ſo many right Ones, excepting 4 therefore the external Ones make thoſe 4.

DEFINITIO XV.

AMong all theſe plain Figures thoſe are call'd *Regular* whoſe Angles and Sides are all equal, as among trilateral Figures the Equilateral Triangle, among Quadrilateral ones, the Square, and in other kinds, ſeveral Species which are not particulariz'd by Names; but all others in whoſe Angles or Sides there is any inequality, are call'd *Irregular*: Tho' ſome of theſe alſo, and all the other may be inſcrib'd in a Circle. Whence you have theſe

CONSECTARYS.

I. THE Areas of the Regular Figures may be obtained yet eaſier, if having found their Center (by *Conſect.* 6. *Definit.* 8.) you draw from thence the Right Lines CB, CA, *&c.* (*Fig.* 32.) till there be form'd as many Triangles ACB, ACF, *&c.* as the Figure has Sides; for ſince all theſe Triangles have their Baſes AB, BF, as ſo many Chords, and their Altitudes CD, CG, as ſo many parts of intercepted Axes DE and GH, and alſo equal *pr. Conſect.* 1. *Definit.* 10. and ſo by *Conſect.* 5. *Definit.* 10. are equal among themſelves; one of their Area's being found and multiplied by the number of Sides, or half the Altitude by the Sum of all the Sides, you'l have the Area of the whole Polygon : For it is manifeſt from what we have already ſaid, and very elegantly Demonſtrated by *F. Pardies, That any Regular Polygon inſcribed in or circumſcribed to a Circle, is equal to the Triangle* Aza, *one Legg whereof is equal to the Perpendicular heighth let fall from the Center upon any Side, and the other to the whole Periphery of the Polygon.* Now if the Triangles into which the Polygon is reſolved, do all ſtand on the ſame Right Line Aa, (*Fig.* 32.) and are all equal and of the ſame heighth, to which the Perpendicular AZ is equal, it will neceſſarily follow, that each pair of Triangles ABZ and ABC, BZF and BCF, *&c.* are equal among themſelves, *pr. Conſect.* 5. *Definit.* 12. and conſequently the Sum of all the former will be equal to the Sum of all the latter, that is, the Triangle Aza to the Polygon given.

II. Since Regular Figures inſcrib'd in a Circle, by biſecting their Arches AB, BF, *&c.* may be eaſily conceived to be changed into others of double the number of Sides, (as a Pentagon into a Decagon, *&c.*) and that *ad Infinitum*; a Circle may be juſtly eſteemed a Polygon of infinite Sides, or conſiſting of an infinite Number of equal Triangles, whoſe common Altitude is the Semidiameter of the Circle : So that the Area of any Circle is equal to a Right Angled Triangle (as AZa) one of whoſe Sides AZ is equal to its (*a*) Semidiameter, and the other Aa to its whole Circumference.

(*a*) *Archimedes of the Dimenſion of the Circle, Prop.* 1.

SCHOLIUM.

IT may not be amiſs to note theſe few things here, concerning the Inſcription of Regular Figures in a Circle.

I. Having deſcribed a Circle on any Semidiameter AC, (*a*) (*Fig.* 33. *N.* 1.) that Semidiameter being placed in the Circumference, will preciſely cut off one ſixth part of it, and ſo become the Side of a Regular Hexagon : and ſo the Triangle ABC will be an Equilateral one, and conſequently the Angle ACB and the Arch AB 60 Degrees, by *Conſ.* 1. *Definit.* 13.

(*a*) *Euclid. Prop.* 15. *lib.* 4.

II. Hence a Right Line AD, omitting one point of the diviſion B, and drawn (*b*) to the next D, gives you the Side of a Regular Triangle inſcrib'd in the Circle, and ſubtends twice 60, *i. e.* 120 Degrees.

(*b*) *Eucl.* 2d *Corol. of the ſame.*

III.

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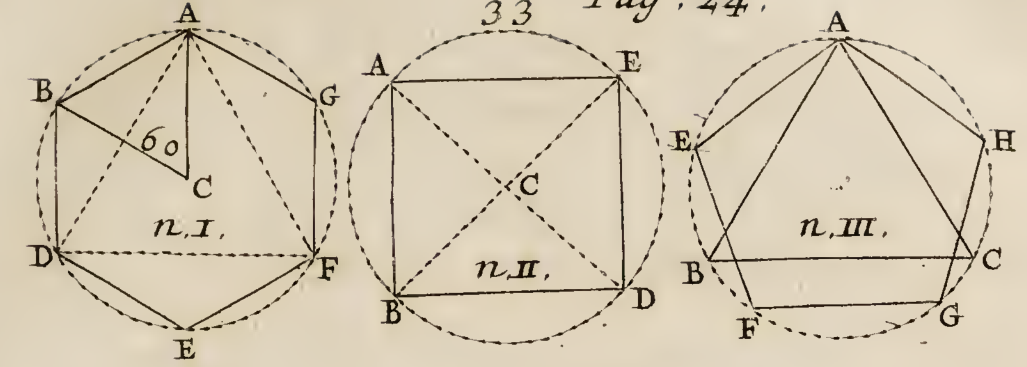

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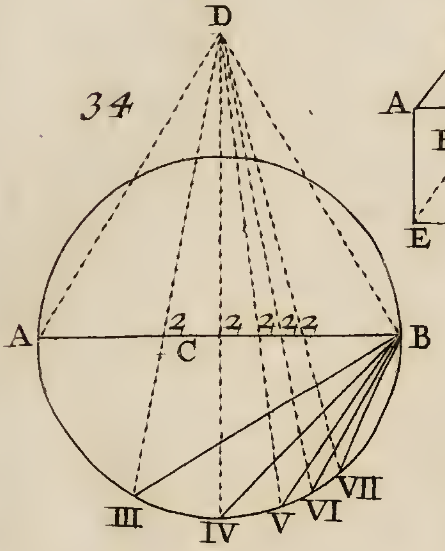

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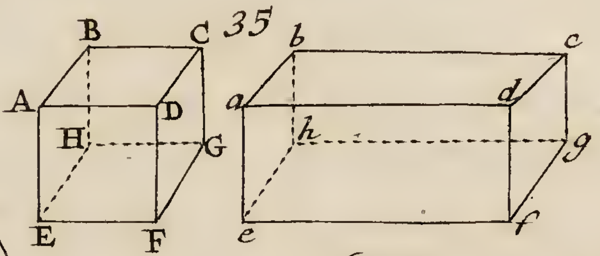

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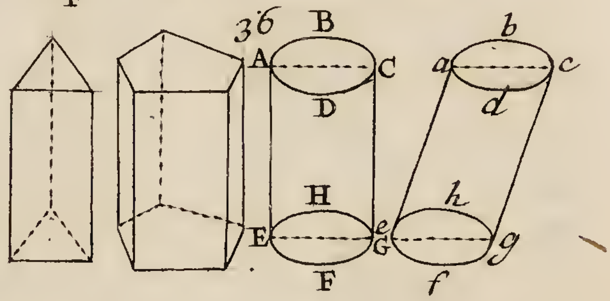

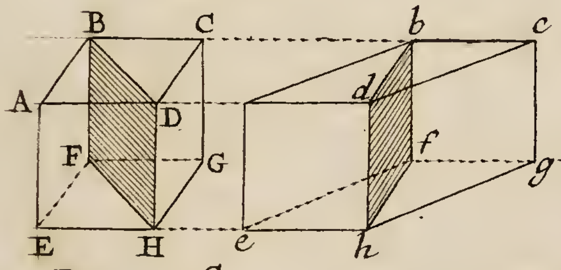

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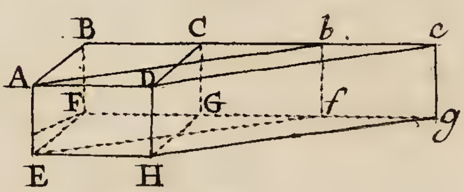

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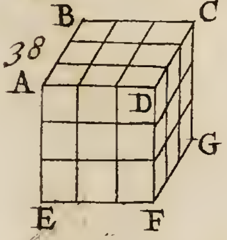

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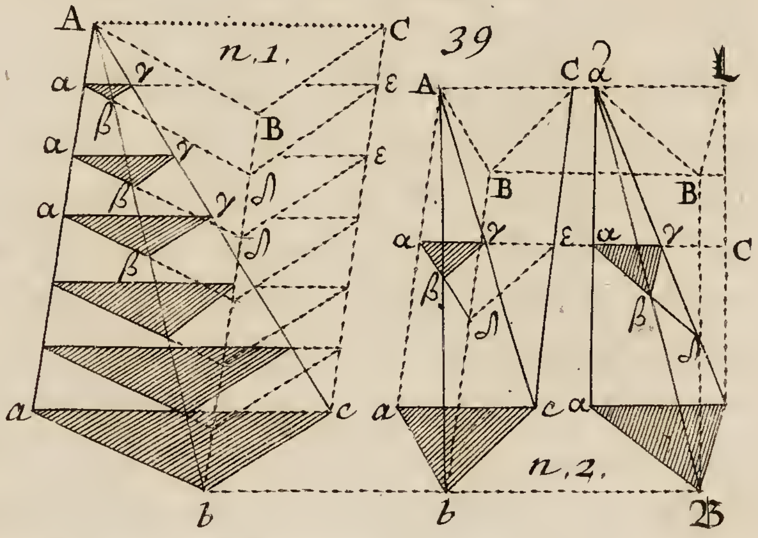

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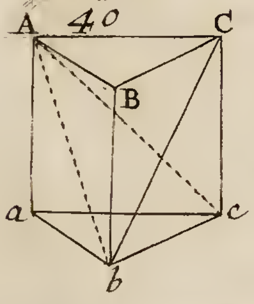


III. If the Diameters of the Circle AD and DE (*N.* 2.) cut one another at Right Angles in the Center C, the Right Lines AB, BD, *&c.* will be the Sides of an inſcribed Square ABDE: For (*c*) the Sides AB, BD, *&c.* are the equal Chords of Quadrants, or Quadrantal Arches, and the Angles ABD, BDE, *&c.* will be all Right ones, as being Angles in a Semicircle (*per Schol.* 6. *Definit.* 10.) compoſed each of two half Right ones, by *Conſect.* 2. *Definit.* 13.

(*c*) *Eucl.* 6. *lib.* 4.

IV. *Euclid* very ingeniouſly ſhews us how to Inſcribe a Regular Pentagon alſo, *Lib.* 4. *Prop.* 10 & 11. and alſo a Quindecagon (or Polygon of 15 Sides) *Prop.* 16. But though the firſt is too far fetch'd to be ſhewn here, yet (ſuppoſing that) the ſecond will eaſily and briefly follow:

In a given Circle from the ſame point A (*N.* 3.) inſcribe a Regular Pentagon AEFGHA, and alſo a Regular Triangle ABC; then will BF be the Side of the *Quindecagon,* or 15 Sided Figure. For the two Arches AE and EF make together 144 Degrees, and AB 120: (*a*) Therefore the difference BF will be 24, which is the 15th. part of the Circumference.

V. The Invention of *Renaldinus* would be very happy, if it could be rightly Demonſtrated; (as he ſuppoſes it to be in his Book of the Circle) which gives an Univerſal Rule of dividing the Periphery of the Circle into any number of equal Parts required, in his 2d Book *De Reſol. & Comp. Mathem.* p. 367. which in ſhort is this: Upon the Diameter of a given Circle AB *Fig.* 34.) make an Equilateral Triangle ABD, and having divided the Diameter AB into as many equal Parts, as you deſign there ſhall be Sides of the Polygon to be Inſcribed, and omitting two, *e. g.* from B to A, draw thro' the beginning of the third from D, a Right Line, to the oppoſite Concave Circumference, and thence another Right Line to the end of the Diameter B, which the two parts you omitted ſhall touch thus, *e. g.* for the Triangle, having divided AB into three equal parts, if omitting the two B2, thro' this beginning of the 3d you draw the Right Line DIII, and thence the Right Line III.B, which will be the Side of the Triangle; and ſo IV.B will be the Side of the Square, VB the Side of the Pentagon, *&c.*

N. B. The Demonſtration of theſe (*Renaldinus* adds, *p.* 368.) we have ſeveral ways proſecuted in our Treatiſe of the Circle: Some of the moſt noted Antient Geometricians, have ſpent a great deal of pains in the Inveſtigation and Effection of this Problem, and ſeveral of the Moderns have loſt both time and pains therein: Whence, we hope, without the imputation of Vain Glory, we may have ſomewhat obliged Poſterity in this point.

DEFINITION VI.

IF the Plane of any Parallelogram AC (*Fig.* 25.) be conceived to move along a Right Line AE, or another Plane AF downwards, remaining always Parallel to its ſelf; there will be generated after this way a Solid having ſix oppoſite Planes Parallel, two whereof, at leaſt, will be equal to one another, whence it is called a *Parallelepiped*; and particularly a *Cube* or *Hexaedrum*, if the *Parallelogram* ABCD that deſcribes it be a Square, and the Line along which it is moved, AE, equal to the Side of that Square, and Perpendicular to the deſcribing Plane, and conſequently all the ſix Parallel Planes comprehending this Solid, equal to one another. But if the deſcribing or Plane Deſcribent (*Fig.* 36.) be a Triangle or Polygon, the Solid is call'd a *Priſm*, if a Circle, it is called a *Cylinder*. Now from the Geneſis of theſe Solids you have the following

CONSECTARYS.

I. IF the Planes or Parallelograms Deſcribent (*a*) ABCD and *a b c d* (*Fig.* 37.) are equal, and their Lines of Motion AE and *a e* alſo equal; the Solids thereby deſcribed, *viz.* Parallelepipeds, Cylinders, and Priſms, (which will therefore have their Baſes and heigths equal) will be equal among themſelves; becauſe the deſcribent Indiviſibles of the one, will exactly anſwer, both in number and poſition, to thoſe of the other, as we have already ſhewn in Parallelograms; Conſequently therefore,

(*a*) *Eucl. l.* 11. p. 29, 30, 31

II. Any Parallelepiped (*b*) may be divided by a Diagonal Plane BDHF (or a Plane paſſing thro' its Diagonals) into two equal Priſms; for by

(*b*) *Eucl. l.* 11. p. 28.

Conſect. 3. *Definit.* 12. the Triangles ABD and BCD, are equal, and are ſuppoſed to be moved by an equal Motion thro' equal ſpaces.

III. And ſince it is evident, even by this Geneſis of them, that in Right Angled Cubes and Parallelepipeds, if the Baſe ABCD (*Fig.* 38.) being divided into little ſquare Area's, be multiplied by the heigth AE, divided by a like meaſure for length, after this way you may conceive as many equal little Cubes to be generated in the whole Solid, as is the number of the little Area's of the Baſe multiplied by the number of Diviſions of the ſide AE; you may moreover obtain the Solidity of any other Parallelepipeds, that are not Right Angled ones, by multiplying their Baſes and Perpendicular Heigths together.

IV. Moreover ſince every Triangular Priſm is the half of a Parallelepiped, and any Multangular Priſm may be reſolved into as many Triangular ones, as its Baſe contains Triangles; you may obtain the Solidity (or Solid Contents) either of the one or the other, if you multiply the Triangular, or Multangular Baſe of them into their Perpendicular Heigth.

V. After the ſame manner you may likewiſe have the Solidity of a Cylinder, which may be conſidered as an Infinite Angled Priſm, juſt as the Circle is as an Infinit-Angled Polygon.

DEFINITION XVII.

IF any Triangle ABC (*Fig.* 39, N. 1.) be conceived to move with one of its Plane Angles C, from the Vertex or top of a Solid Angle (determined by two Planes aAb and cAa joined together in the common Line Aa) with a motion always parallel to it ſelf; ſo that its extreme Angular Point A ſhall always remain in the Line Aa, but with its Sides AB and AC ſhall all along raze on the two Angular Planes, till at length it falls wholly within the Solid Angle: by this its motion it will deſcribe within the Solid Angle, the Figure we call *Pyramidal*, whoſe Baſe will be the Triangle *a b c*, and its Vertex A will alſo deſcribe without it another Quadrangular Pyramid, whoſe common Vertex will be the ſame A, but the Baſe the Quadrangle Cb, deſcribed by the Side of the moveable Triangle BC: The firſt Pyramid it will deſcribe with its Triangular Parts, *αβγ* continually increaſing from the point A, and ending in the Triangle *a b c*; but the latter Pyramid will be deſcribed by the remaining parts, continually decreaſing downwards from the whole ABC, and the Quadrangular Trapeiza Brδe at length ending in the Right Line *b c*: *S*o that in the mean while the ſaid Triangle with its whole ſpace deſcribes, according to what we have ſaid before, the Triangular Priſm compoſed of thoſe two Pyramids. From this Geeeſis of Pyramids you'l have the following

CONSECTARYS.

I. OF what ſort ſoever the Deſcribing Triangles are ABC, 𝔄𝔅ℭ (N. 2.) ſo they are equal; and whatever the Solid Angles are, comprehended under the Planes abA, and acA, 𝔞𝔟𝔄 and 𝔞𝔠𝔄, ſo they are accommodated to the Plane Angles A and 𝔄, and ſuch that, *&c.*

DEFINITION XVIII.

THere may be exhibited another eaſier Geneſis of Cones and Pyramids,but it reſpects only the Dimenſion of the Surface, and not of the *S*olidity of them, *viz.* If you have a fix'd point A that is not in the Angular Plane BCDEF (*Fig.*41.) and a Rght Line AF let fall from that point to any Angle of the Plane,be conceived to move round the ſides BC, CD, *&c.* This Plane by its motion will deſcribe as many Triangles ABC, CAD, *&c.* as the Angular Plane has Sides. And theſe Triangles all meeting at the point A, make that Solid which we call a *Pyramid.* Now if inſtead of the Angular Plane there be ſuppoſed a Circular one, (or an Angular one of Infinite *S*ides) the Solid thence produc'd is called a *Cone*, whoſe Surface is equal to Infinite Triangles, conſtituted on the Baſe BCDE, and whoſe *S*olidity would conſequently equal an Infinite Angled Pyramid of the ſame heigth. And after the ſame manner by the motion of the Line AF, remaining always Parallel to it ſelf about Parallelograms or Triangular Planes, will be generated Parallelipipeds, Priſms, and Cylinders. But as one Pyramid will be produced more upright than another, according as the point A ſtands more over the middle of the Plane BCD, *&c.* (*Fig.* 42.) or reſpects it more obliquely: So in particular a *Cone* is called a *right Cone*, when the line AO being let fall to the Center of the circular Plane (otherwiſe call'd the Axis of the Cone) conſtitutes on all ſides right Angles with it; but it is call'd an *Oblique* or *Scalene* Cone, when the Ax ſtands obliquely on the Baſe. Which diſtinction may alſo be eaſily underſtood when apply'd to Cylinders, tho' a right Cone and Cylinder may alſo be conceiv'd to be generated after another way, as if for the one a Triangle and for the other a right angled Parallelogram AOB and LAOB be be conceiv'd to be moved round a line AO conſidered as immoveable (whence it is call'd an Ax;) and alſo a truncated Cone may be formed if a right angled Trapezium, 2 of whoſe Sides are parallel be moved, &c. And as we have deduc'd the Solidity of theſe Bodies from the foregoing Geneſis, ſo their external Surfaces, as alſo of Priſms and Parallelepipeds may eaſily be found from the preſent Geneſis, by any one who attentively conſiders the following

(a) *Archimedes lib.* 1 *de Con. and Cylin. prop.* 7. *and* 8.

COROLLARYS.

I. SInce the whole external Surface, except the Baſe, of any Pyramid is nothing but a Syſtem of as many Triangles ABC, CAD &c. as the Pyramid has Sides; if the Area's of thoſe Triangles ſeparately found by *Conſect.* 8. *Def.* 12. be added into one Sum, you'l have the Superficial Area of the whole Pyramid.

II. If a Pyramid be cut with a Plane *b*, *c*, *d*, *e*, parallel to its Baſe BCDE (*Fig.* 41.) The Surface of that truncated Pyramid comprehended between the prallel Planes may be obtain'd if having found the Surface of the Pyramid A*b c d e* cut off from the reſt by *Conſect.* 1. you ſubtract it from the Surface of the whole Pyramid.

II. The external Surface of a right Pyramid that ſtands on a regular Polygon Baſe is equal to a Triangle, whoſe Altitude is equal to the Altitude of one of the Triangles which compoſe it, and its Baſe to the whole Circumference of the Baſe of the Pyramid.

IV. Therefore the Surface of a right Cone, by what we have already ſaid, is equal to a Triangle whoſe heighth is the ſide of the Cone, and the Baſe equal to the Circumference of the Baſe of the Cone.

V. The Surface of a truncated right Cone, or Pyramid is equal to a Trapezium which has 2 parallel Sides the loweſt of which is equal to the Perifery of the Baſe, and the other to the Perifery of the Top or upper Part, and the heigth, to the intercepted Part.

VI. The Surface of a right Cylinder or Priſm is equal to a Parallelogram which has the ſame height with them, and for its Baſe a right line equal to the Perifery of that Cylinder or Priſm.

DEFINITION XIX.

IF a ſemicircular Plane ADB (*Fig.* 43. N. 1.) be conceived to move round its Diameter AB which is fixt, as an Axis, by this Motion it will deſcribe a ***Sphere***, and with its Semicircumference the Surface of that *Sphere*; every part whereof is equally diſtant from the middle Point of that Axis C (which is therefore call'd the Center of that *Sphere*.) Now if (N. 2.) this ſemicircular *Ambitus* (a) be conceived to be divided before that revolution, firſt into 2 Quadrants AD, and BD, and then each of thoſe again into as many parts equal in Number and Magnitude as you pleaſe, and having drawn the Chords AF, FE, ED, &c. let the Polygon AFEDGHB Inſcribed in the Semicircle be conceived together with it to be turn'd about the Axis AB; then will A 1 F, and B 4 F deſcribe 2 Cones about the Diameters F f, and H h; and the Trapezia abont the Axes 1, 2, 2 C, C 3, and 34, will deſcribe ſo many truncated Cones, and the lines AF, FE, &c. ſo many Conical Superficies, by the Antecedent *Def.* and ſo the whole Polygonal Plane AFEDGHB a Conical Body inſcribed in the Sphere, and contain'd under only Conical Surfaces. And as any attentive Perſon may eaſily perceive ſuch a Body to be leſs than the ambient Sphere, and its whole Surface leſs than the Surface of the ambient Sphere; ſo he may as eaſily trace theſe following

(a) *Archimedes lib.* 1. *de Cono & Cyl. Prop.* 22. *Coroll. & Prop.* 23.

CONSECTAYS.

I. IF the Arches AF, FE, *&c.* be further biſected, and a Polygonal Figure of double the number of Sides inſcribed in the Semicircle, and conceived to be moved round after the way we have already ſhewn, the Pſeudoconical Body hence ariſing, will approach nearer and nearerto the Solidity of the Sphere, and the Surface of the one to the Surface of the other, and hence (if we continue this Biſection, or conceive it to be continued *ad Infinitum*) you may infer.

II. That a Sphere may be look'd upon as much a Pſeudoconical Body, conſiſting of infinite Sides, and it's Surface will be equal to the infinite Conical Surfaces of that Body; which we will take further notice of below.

DEFINITION XX.

IF the Diameter AB of the Semicircular Plane ADB (*Fig.* 43. N. 3.) be conceived to be divided into equal Parts (as here the Semidiameter AC into 3.) and if the circumſcribing Parallelograms CE, 2 E, 1 G on the tranſverſe Parallels CD, 2 *e*, 1 *f* be conceived together with the Semicircle it ſelf to revolve about the fixed Ax AB; it is evident that there will be formed from the Semicircle a Sphere as before, and from the Circumſcribed Parallelograms, ſo many circumſcribed Cylinders of equal heighth: but if all the Altitudes or Heighths of theſe are biſected or divided into two, and ſo make the number of circumſcribing Parallelograms double, there will be formed (by moving them round as before) double the Number of Cylinders of half the heighth, but which yet being taken together, approach much nearer the ſolidity and roundneſs of the Sphere, than the former; which were fewer in Number (*viz.* the ſix latter Parallelograms approach nearer to the Plane of the Circle than the three former) and thus if that biſection of the Altitudes be conceived to be continued *ad infinitum*, the innumerable Number of thoſe infinitely little Cylinders will coincide with the Sphere it ſelf. Moreover if you conceive any Polyedrous or Multilateral Figure to be circumſcribed about the Sphere (which we here endeavour to delineate by the Polygon ABCD N. 4. circumſcribed about the Circle)

cle) and the ſolid Angles thereof to be cut by other Planes *ab*, which ſhall touch the Sphere ; it is manifeſt there will thence ariſe another Polyedrous Figure, the Solidity whereof will approach nearer to the Solidity of the Sphere, and its Surface to the Spherical Surface than the former , and if the Angles of this be again in like manner cut off, there will ſtill ariſe another new Solid , and new Surface approaching yet nearer to the Solidity and Surface of the Sphere than the former, *&c.* and ſo after an infinite Proceſs they will coincide with the Sphere and its Surface themſelves. Whence there flow theſe

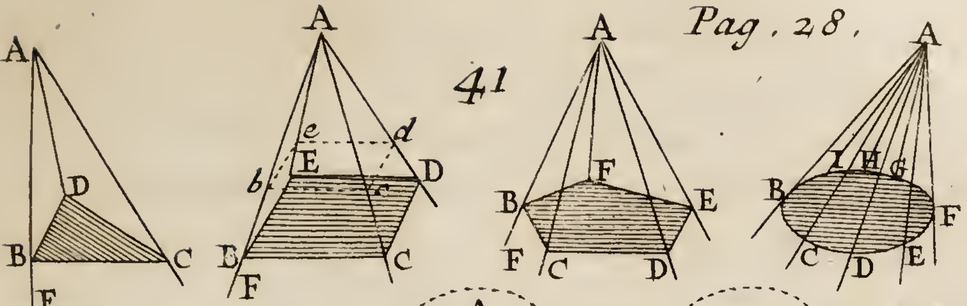
COROLLARYS.

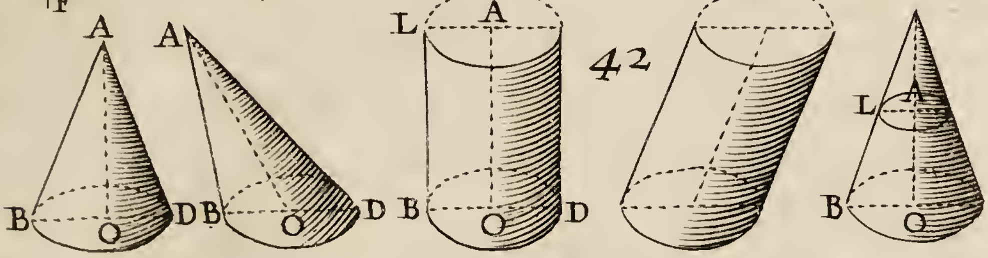
I. THE Sphere may be conſidered as a Polyedrous Figure, or as conſiſting of innumerable Baſes, *i. e.* compoſed of an innumerable Number of Pyramids, all whoſe Vertex's meet in the Center, and ſo whoſe common heigth is the Semidiameter of the Sphere, and the ſum of all the Baſes equal to the Superficies of the Sphere.

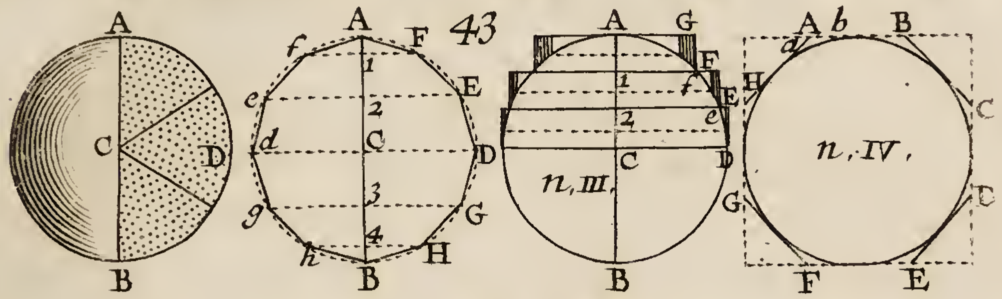
II. If you can find a Proportion between a Cylinder of the ſame heigth with any Sphere, and whoſe Baſe is equal to the greateſt Circle of that Sphere, and innumerable Cylinders circumſcribed about it, as we have juſt now ſhewn ; then you may alſo obtain the Proportion between the ſaid circumſcrib'd Cylinder and the inſcrib'd Sphere : Which to have here hinted may be of ſervice hereafter in its proper place.

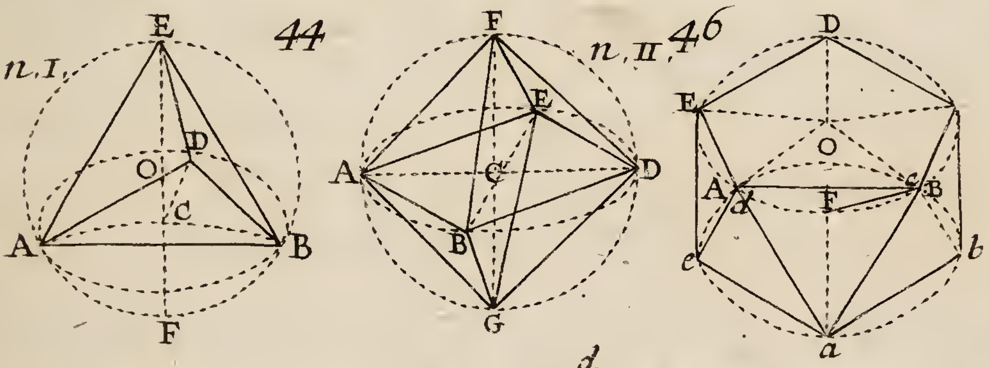
DEFINITION XXI.

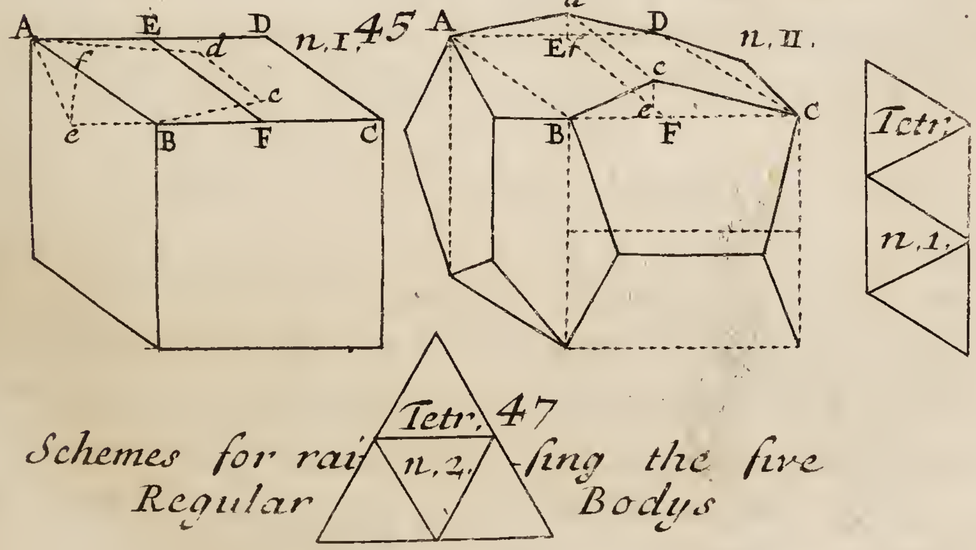
THere remain thoſe Bodies to be conſider'd which are call'd *Regular*, which correſpond to the Regular Plane Figures ; and as thoſe conſiſt of equal Lines and Angles, ſo theſe likewiſe are comprehended under Regular and Equal Planes meeting in equal ſolid Angles ; and as thoſe may be Inſcribed and Circumſcribed about a Circle, ſo may the latter likewiſe in and about a Sphere. But whereas there are infinite Species of Regular Plane Figures, there are only five of Regular Solids ; the firſt whereof is contained under four Equal and Equilateral Triangles, whence it is nam'd a *Tetraedrum* ; the ſecond is terminated by ſix equal Squares, and thence is call'd *Hexaedrum*, and otherwiſe











Schemes for raiſing the five Regular Bodys

wiſe a *Cube*; the third being comprehended under eight Equal and Equilateral Triangles, is call'd an *Octaëdrum*; the fourth is contained under twelve Regular and Equal Pentagons, and ſo is nam'd a *Dodecaëdrum*; the fifth, laſtly, is contained under twenty Regular and Equal Triangles, and is thence nominated an *Icoſaëdrum.* Beſides theſe five ſorts of Regular Bodies there can be no other; for from the concourſe of three Equilateral Triangles ariſes the Solid Angle of a *Tetraëdrum*, from four the Solid Angle of an *Octaëdrum*, from five the Solid Angle of an *Icoſaëdrum*; from the concourſe of four Squares you have the Solid Angle of an *Hexaëdrum*; from that of three Pentagons you have the Solid Angle of a *Dodecaëdrum*; and in all this Collection of Plane Angles, the Sum does not ariſe ſo high as to four Right ones. But four Squares, or three Hexagons meeting in one Point, make preciſely four Right Angles, and ſo by *Conſect.* 2. *Definit* 8. would conſtitute a Plane Surface, and not a Solid Angle. Much leſs therefore could three Heptagons or Octagons, or four Pentagons meet in a Solid Angle, to form a new Regular Body; for thoſe added together would be greater than four Right Angles. But now, for the Meaſures of theſe five Regular Bodies, take the three following

CONSECTARYS.

I. Since a Tetraëdrum is nothing elſe but a Triangular Pyramid, and an Octaëdrum a double Quadrangular one, their Dimenſion is the ſame as of the Pyramids in *Schol.* of *Definit.* 17.

II. The Solidity of an Hexaëdrum may be had from *Conſect.* 3. *Definit.* 13.

III. A Dodecaëdrum conſiſts of twelve Quinquangular Pyramids, and an Icoſaëdrum of twenty Triangular ones, all the Vertex's or tops whereof meet in the Center of a Sphere that is conceived to circumſcribe the reſpective Solids, and conſequently they have their Altitudes and Baſes equal: Wherefore having found the Solidity of one of thoſe Pyramids, and multiplied it by the number of Baſes (in the one Solid 12, in the other 20) you have the Solidity of the whole reſpective Solids.

DEFINITION XXII.

BEſides theſe Definitions of the Regular Bodies, we may alſo form like Idea's of them from their Geneſis, which particularly *Honoratus Fabri* has given us a ſhort and ingenious Syſtem of, in his *Synopſis Geometrica*, p. 149. and the following.

I. Suppoſe an Equilateral Triangle ABD to be inſcrib'd in a Circle (*Fig.* 44. *N.* 1.) whoſe Center is C, whence having conceived the Radii CA, CB, CD, to be drawn, imagine them to be lifted up together with the common Center C, ſo that the point C aſcending Perpendicularly, at length you'l have the Lines EA, EB, ED, equal to the Lines AB, BD, DA, After this way there will be generated, or made a Space conſiſting of four Equal and Equilateral Triangles, which is call'd a *Tetraëdrum.* Hence we ſhall by and by eaſily demonſtrate, the quantity of the Elevation CE, and the Proportion of the Diameter of the Sphere EF to be Circumſcribed to the remaining part CF, and ſo the reaſon of the Euclidean Geneſis propoſed *lib.* 13. *Prop.* 13.

II. Much like this, but ſomewat eaſier to be conceived, is the Geneſis of the *Octaëdrum*, where by a mental raiſing of the Center C (*Fig.* 44. *N.* 2) of the Square ABDE inſcribed in the Circle, together with the Semidiameters CA, CB, CD, CE, until being more and more extended they at length become the Lines AF, BF, DF, EF, all equal among themſelves, and to the ſide of the Square AB or BD; and its manifeſt, that by the like extenſion conceived to be made downwards to G, there will be formed eight equal and regular Triangles, which will all concur in the two oppoſite Points F and G. We might alſo deduce another Geneſis of the Octaëdrum from a certain Section of a Sphere, and alſo give the like of a Hexaëdrum or Cube: but we have already given the eaſieſt, of the one, *viz.* that which is alſo common to Parallelepipeds; and that of the other juſt now given is ſufficient to our purpoſe.

CHAP.

CHAP. II.

Containing the Explication of thoſe terms, which relate to the affections of the Objects of the Mathematicks.

DEFINITION XXIII.

Every Magnitude is ſaid to be either *Finite* if it has any bounds or terms of its Quantity; or *Infinite* if it has none, or at leaſt *Indefinite* if thoſe bounds are not determined, or at leaſt not conſidered as ſo; as *Euclid* often ſuppoſes an *Infinite* Line, or rather perhaps, an *Indefinite* one, *i. e.* conſidered without any relation to its bounds or Ends: By a like diſtinction, and in reality the ſame with the former, all quantity is either *Meaſurable*, or ſuch that ſome Meaſure or other repeated ſome number of Times, either exactly meaſures and ſo equals it, (which *Euclid* and other *Geometricians* emphatically or particularly call Meaſuring) or elſe is greater; or on the other ſide *Immenſe*, whoſe Amplitude or Extenſion no *Finite* Meaſure whatſoever, or how many times ſoever repeated, can ever equal: In the firſt Caſe, on the one Hand, the Meaſure (*viz.* which exactly meaſures any quantity) is called by *Euclid* an *aliquot Part* (*a*) or ſimply a *Part* of the thing meaſured: as *e. g.* the Length of one Foot is an aliquot Part of a Length or Line of 10 Foot. In the latter Caſe the Meaſure (which does not exactly meaſure any Quantity) is called an *Aliquant* Part, as a line of 3 or 4 Foot is an *Aliquant* part of a Line of 10 Foot. Now therefore, omitting that perplext Queſtion, whether or not there may be an infinite Magnitude, we ſhall here, reſpecting what is to our purpoſe, deduce the following

(*a*) *lib.* 5. *Def.* 1.

CONSECTARY.

Every Meaſure, or part ſtrictly ſo taken, is to the thing Meaſured, or its whole, as Unity to a whole number, for that (which is one) repeated a certain number of times, is ſuppoſed exactly to meaſure the other.

DEFINITION XXIV.

IF the ſame Meaſure meaſures 2 different quantities (whethe the one can exactly Meaſure the other or not) thoſ Quantities are ſaid to be abſolutely *Commenſurable*; but if the can have no common Meaſure; they are called *Incommenſurable* Notwithſtanding which they both retain one to the other a certain relation of Quantity, which is call'd *Reaſon* or *Proportion*, a we ſhall further ſhew hereafter. In the mean while we hav hence, as an infallible Rule to try whether Quantities can admit o a Common Meaſure or not, this

CONSECTARY.

THoſe Quantities are Commenſurable, whereof (*a*) one i to the other, either as Unity to an whole Number, or a one whole Number to another, for either one o them is the Meaſure of the other, as alſo of i ſelf, and then it is to that other as Unity to Number by the *Conſect.* of the *preced.* or elſ they admit of ſome third Quantity for a common Meaſure which will be to either of them ſeparately as Unity to ſome Number: therefore they are one to another as Number to Number.

(*a*) *Euclid lib.* 10. *Prop.* 5, 6, 7, 8.

DEFINITION XXV.

IF 2 Quantities of the ſame Kind, conſidered as Meaſures one of the other, being applyed one to the other, exactly agree, or are exactly equal every way, (as *e. g.* 2 Squares on the ſame common Side, or two Triangles whoſe Lines Angles and Spaces exactly agree and conicide) or at leaſt may be equally meaſured by a common Meaſure applyed to both) as *e. g.* a Square and an Oblong, or a Rhombus, or Triangle, each of whoſe Area's were 20 ſquare Inches, altho' they do not agree in Lines and Angles; the firſt may be called *Simply Equal*, and the other *totally equal, or equal as to their wholes*: But if one be greater and the other leſs, they are *Unequal*, and that which exceeds is called the *greater*, and that which is deficient the *leſs*, and that part by which the leſs is exceeded by the greater, in reſpect to the greater is call'd *Exceſs*, in reſpect to the leſs *Defect*, and by a common Name they are call'd the *Difference*. All which as they are plain and eaſy, ſo they afford us a great many ſelf-evident Truths, which are uſed to be call'd Axioms, as theſe and the like

CONSECTARYS.

I. THe whole is greater than its Part, whether it be an Aliquot or aliquant Part.

II. Thoſe Quantities which are equal to a third are equal betwixt themſelves.

III. That which is greater or leſs than one of the equal Quantities is alſo greater or leſs than the other.

IV. Thoſe Quantities which, being applyed one to the other, or placed one upon the other, either really or mentally, agree; may be eſteemed as *totally equal*: And on the Contrary,

V. Thoſe Quantities which are *totally equal* will agree, *&c.* To which might be added ſeveral others which we have already made uſe of and ſuppoſed as ſuch in the preceding *Definitions.*

DEFINITION XXVI.

THere are moreover *Addition*, *Subtraction*, *Multiplication* and *Diviſion*, which are common affections of all Quantities as well as of Numbers. *Addition* is the Collection of ſeveral Quantities (for the moſt part of one kind) into one total or Sum; which is either done ſo, that the whole (which is commonly called the *Sum* or *Aggregate*) obtains a new Name, or elſe by a bare connexion of the Quantities to be added by the Copulative *and*, or the uſual Sign + (*i. e. plus* or more) as for Example 2 Numbers . . . and : . . . (ſuppoſe 3 and 4) added together make the Sum . . . . ; (*i. e.* 7, or which is the ſame thing 3+4;) and this Line ——— added to this other ——— gives the Sum ———— which is nothing but the 2 Lines joyn'd, or taken together. But now if we would treat of theſe Lines, or any other 2 Quantities to be added, more generally; by calling the firſt a (a) and the latter (b) we may fitly write their Sum $a+b$.

SCHOLIUM.

HAving thus explained the Term of *Addition*, theſe and the like Axioms emerge of themſelves: *If to equal Quantities you add Equal the Sum will be Equal; but if to Equal you add unequal the Aggregate will be unequal;* &c. Moreover it may not be amiſs to admoniſh the *Tyro* of theſe 2 things. 1. In Addition may be ſeen the vaſt uſefulneſs of that very Ingenious tho' familiar Invention, mentioned in *Definit.* 3. for hereby we may collect into one Sum not only Tens, and Hundreds, but Thouſands, Millions, Myriads, as tho' they were only Units; which we will Illuſtrate by an Example.

DEFINITION XXVII.

SUbtraction is the taking one Quantity from another (of the ſame kind;) which is ſo performed that either the remainder obtains a new Name, or by a bare ſeparation of the *S*ubtrahend by the privative Particle leſs, or the uſual *S*ign — which ſtands for it, as *e. g.* . . . or three being ſubtracted from . . . . or 7, the remainder or difference is . . . . or 4 and this Line ———— *S*ubtracted from that —————— leaves ———— Now if we would ſignify this more generally either of theſe Lines, or the Number above, or any 2 Quantities whatſoever that are to be Subtracted one from the other, by naming the firſt (a) and the latter (b) we ſhall have the remainder $a - b$. Herein are evident theſe and the like Axioms: *If from equal Quantities you Subtract Equal ones, the Remainders or Differences will be equal.* Here it will be worth while to take notice of, from this and the *preced. Definit.* the following

CONSECTARYS.

I. IF a negative Quantity be added to it ſelf conſidered as poſitive (as -3 to $+3$ or $-a$ to $+a$) the Sum will be *o*; for to add a Privation or Negative is the ſame thing as to Subtract a Poſitive, wherefore to join a Negative and Poſitive together, is to make the one to deſtroy the other.

II. If a negative be ſubtracted from its poſitive ($-a$ from $+a$) the remainder will be double of that poſitive ($+2a$) for to ſubtract or take away a privation or negative, is to add that very thing, the privation of which you take away; for really that which in words is called the *addition of a Privation*, is in reality a Subtraction, and a *ſubtraction* of it, is really an *addition*; and what is here call'd a *Remainder*, is indeed a Sum or Aggregate; and what is there call'd a *Sum*, is truly a *Remainder*. Thus,

III. If the poſitive Quantity ($+a$) be taken from the privative one ($-a$) the remainder is double the privative one ($-2a$) ſince, taking away a poſitive one, there neceſſarily ariſes a new Privation which will double that you had before. Hence,

IV. You have the Original of the Vulgar Rules in Literal Addition and Subtraction: *If the Signs of the unequal Quantities are different, in the room of Addition you muſt ſubtract, and in room of Subtraction add, and to the ſum or remainder, prefix the Sign in the firſt place of the greateſt, in the next of that from which you Subtract: but if the Signs are both the ſame, and the greateſt quantity to be ſubtracted from the leſs, you muſt, on the contrary, ſubtract according to the natural Way, the leaſt from the greater, and prefix the contrary Sign to the Remainder:* Which Rules you may ſee Illuſtrated in the following Examples:

Addition	*Subtraction.*	
$4b-2a$	from $2a+b$	from $3a+2b$
$3b+5a$	Subſt. $a-b$	Subſt. $2a+3b$
$7b+3a$	R. $a+2b$	R. $a-b$

NOTE.

☞ Inſtead of the Authors 4th Conſect. as far as it relates to Subtraction, which may ſeem a little perplext, take this general Rule for Subtraction in Species, *viz.* Change all the Signs of the lower Line, or Subtrahend, and then add the Quantities, and you have the true Remainder.

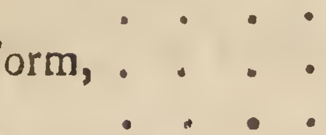
SCHOLIUM.

IN this Literal Subtraction, we have not that conveniency which the invention of Vulgar Notes ſupplies us with, that from the next foregoing Note we may borrow *Unity*, which in the following Series goes for 10, *&c.* This is done in Tetractycal Subtraction only with this difference, that an Unite here borrowed goes only for 4. That the eaſineſs of this Operation may appear, we will add one Example, wherein from
this number, —— 1232002310232
you are to ſubtract this, 321012321223

310323323003

Whereever therefore the inferiour Note is greater than the ſuperiour one, the facility is much greater here than in common Subtraction, becauſe never a greater number than 3 is to be ſubtracted out of a greater, than 4 and 2: but if the inferiour number be greater than the ſuperiour, you borrow unity from the left hand, which is equivalent to 4; the reſt is perform'd as in common Subtraction.

DEFINITION XXVIII.

MUltiplication, generally Speaking, is nothing elſe but a Complex or manifold Addition of the ſame quantity, wherein that which is produced is peculiarly call'd the *Product*, and thoſe quantities by which it is produced, are called the *Multiplicand* and the *Multiplier*: The firſt denotes the Quantity which is to be multiplied, or added ſo many times to its ſelf; and the other the Number by which it is to be multiplied, or determins how many times it is to be added to it ſelf. The ſame terms are applyed moreover to Lines and other Quantities. But here are two things to be chiefly noted; 1. That the Multiplication of one number by another, or of a Line by a Line, may be conſidered as having a double Event; for the Product may be either of the ſame or a different kind, as, *e. g.* when . . . . 4 is multiplied by 3 . . . the product may be conſidered either as a Line, thus, ... or as a Plane Surface in this Form,  Whence it is alſo named a Plane Number, and the product is conceived to be formed by the motion of an erect Line AB, conſiſting of 3 equal parts, along another BC, conſiſting of 4 equal parts, and conceived as lying along. So alſo the Multiplication of Lines (*e.g.* of the Line A ——— B by the Line B———————C) may be conceived to be ſo performed, that the Product alſo ſhall be a Line, *e. g.* C———————————————D (concerning the uſefulneſs of which Multiplication in *Geometry*, we ſhall have occaſion to ſpeak more hereafter;) or ſo, that the Product ſhall be a Plane or Surface, ariſing from the motion of the erect Line AB, along AC, conceiv'd as lying along; as we have already ſhewn. But as for the moſt part theſe Planes ſo produced are called *Rectangles*, if the Lines that form them are unequal; but if they are equal they are call'd *Squares*, (otherwiſe the Powers of the given Quantities;) and in this caſe the Lines that form them are called *Square Roots*; ſo alſo if thoſe Planes are multiplied again into a third Quantity (as either a Line or a Number) there will ariſe *Solids*, and particularly if that third Quantity be the Root of the Square, the Product is called a *Cube*, &c. The other thing to be noted is, That both theſe ways of Multiplying either Numbers or Lines, are expreſſed by a very compendious, tho arbitrary way, of Notation, *viz.* by a bare *Juxtapoſition* of the Letters which denote ſuch and ſuch Species of Quantities, as, *e. g.* if for the forementioned Number or Line AB we put a, and for BC b, the Product will be ab; or if the Efficients are equal, as a and a the Square thence produced, will be aa or a^2; and if this Square be further multiplied by its Root a, then the Cube thence produced will be aaa or a^3, &c. Which being premiſed, you have theſe following

CONSECTARIES.

I. IF a Poſitive Quantity be multiplied by a Poſitive one, the Product will be alſo Poſitive; ſince to multiply is to repeat the Quantity according as the Multiplier directs: Wherefore to multiply by a Poſitive Quantity, is to repeat the Quantities poſitively; as on the other ſide, to multiply by

a

a Privative, is ſo many times to repeat the Privation of that Thing: Which we ſhall ſhew further hereafter.

II. Equal Quantities (a and a) multiplied by the ſame (b), or contrariwiſe, will give equal Products (ab and ab or ba).

III. The ſame Quantity (z) multiplied by the whole Quantity ($a+b+c$) or by (*a*) all its parts ſeparately, will give equal Products. Alſo

IV. The whole ($a+b$) whether it be multiplied by (*b*) it ſelf, or by its parts ſeparately, will give equal Products.

(*a*) *Eucl. lib.* 2. *prop.* 1.
(*b*) *lib.* 2. *prop.* 2.

SCHOLIUM I.

THe Vulgar Praxis of Numeral Multiplication, is founded on theſe two laſt Conſectarys, as *e. g.* to multiply 126 by 3; you firſt multiply 6 by 3, then 2, *i. e.* 20 by 3, then 1, *i. e.* 100 by the ſame, and then add each of thoſe partial Products into one Sum: In like manner being to multiply 348 by 23, you firſt multiply each Note of the Multiplicand by the firſt of the Multiplier (3) and then by the ſecond (2) (*i. e.* 20) &c. which is to be done likewiſe after the ſame manner in Tetractical Multiplication; only in this latter, which is more eaſie, you have nothing to reſerve in your mind, but all is immediately writ down, (which might alſo be done in Vulgar Multiplication) as may be ſeen by this Example underneath, as alſo the great eaſineſs of this ſort of Multiplication, beyond the common way, becauſe there is no need of any longer Table than that we have ſhewn page 7.

```
      3 2 1 0 3 3
            1 2 3
  ---------------
    2 1 2 3 2 1 2
    1 1 2 1 0 1
  1 2 0   0 1 0
  3 2 1 0 3 2
            2
-----------------
1 2 0 0 2 0 3 2 2
```

SCHOLIUM II.

It is manifeſt from what we have ſaid,

I. If the Baſe of a Parallelogram be called (b) and its Altitude a, its Area may be expreſſed by the Product ab, by *Conſ.* 7. *Definit.* 12.

II. If the Baſe of a △ be b or eb, and its Altitude a its Area will be half ab or half eab, by *Conſectary* 8. of the ſame *Definition.*

III. If the Baſe of a Priſm or Parallelepiped or Pyramid be half ab or ab, and its Altitude c, the ſolid Contents of that Priſm will be half abc, and of the Parallelepiped abc, by *Conſect.* 3 & 4. *Def.* 16. and of the Pyramid $\frac{1}{6}abc$, by *Conſ.* 3. *Def.* 17.

DEFINITION XXIX.

Diviſion, in general, is a manifold or complicated Subtraction of one quantity (which is called the *Diviſor*) out of another (which is called the *Dividend*) whoſe multiplicity, or how many times the one is contained in the other, is ſhewn by another quantity ariſing from that Diviſion, which is therefore called the *Quote* or *Quotient.* Here alſo the *Diviſor* is of the ſame kind with the *Dividend*, or of a different kind, *e. g.* of the ſame kind if the product . . . . (12) be divided by (3) whence you'l have the Quotient . . . . (4) or dividing the aforementioned Line CD by the Line AB you'l again have the Line BC; but of a different kind, if the plane number a-

. . . .
bove found . . . . or the Rectangle ABCD be divided by
. . . .

a Retroduction, or a moving backwards again the erect Side AB, by whoſe motion the Rectangle was firſt formed, that ſo the Line BC may remain alone again. But both theſe kinds of Diviſion as they have their peculiar Difficulties in Arithmetick and Geometry, which we ſhall further elucidate in their proper places; ſo they may be univerſally and very eaſily performed in Species (or by Letters) which will be ſufficient to our preſent purpoſe; or by a bare ſeparation of the Diviſor from the Dividend, if it be actually therein included; or by placing the Diviſor underneath the Dividend with a Line between. Thus if *a b* be to be divided by (*b*) the Quotient will be *a*; if by *a*, the Quotient will be (*b*); but if *a* or *a b* be to be divided by *c* which Letter ſince it is not found in the Dividend, cannot be taken out of it) the Quotients are $\frac{a}{c}$ and $\frac{ab}{c}$ *i. e. a* or *a b* divided by *c*, after the ſame manner as if 2 were to be divided by 3; which Diviſor, ſince it is not contained in the Dividend, is uſually placed under it, by a ſeparating Line thus, $\frac{2}{3}$, 2 divided by 3.

SCHOLION.

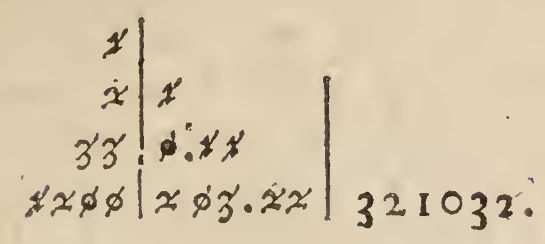
HOW difficult Common Diviſion is, eſpecially of a large Dividend by a large Diviſor, is ſufficiently known: but how eaſily it is performed by Tetractical Arithmetick, we will barely bring one Example to ſhew. If the Product found in *Schol.* 1. of the preceding *Definition*, 1200 203 22 be again to be divided by its Multiplier 133, it may be performed after the uſual way, but with much more eaſe, as the following Operation will ſhew; or according to a particular way of *Weigelius*, by writing down the Diviſor, and its double and triple, in a piece of paper by it ſelf, after this way:

123	312	1101
Diviſor,	*Double,*	*Triple.*

and then moving that piece of Paper to the Dividend, note, which of thoſe three Numbers comes neareſt to the firſt Figures of the Dividend; for that barely ſubtracted gives the Remainder, and will denote the Quotient to be writ down in its proper place; as the operation itſelf will ſhew better than any words can.

$$\begin{array}{c}
\cancel{1}\cancel{1}\cancel{1}\cancel{1}\cancel{1} \\
\cancel{1}\cancel{3}\cancel{2}\cancel{2}\cancel{2}\cancel{2} \\
\cancel{3}\cancel{2}\cancel{3}\cancel{0}\cancel{1}\cancel{1}\cancel{1} \\
\cancel{1}\cancel{2}\cancel{0}\cancel{0}\cancel{2}\cancel{0}\cancel{3}\cancel{2}\cancel{2} \\
\cancel{1}\cancel{2}\cancel{3}\cancel{3}\cancel{3}333 \\
\cancel{1}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}2 \\
\cancel{1}\cancel{1}\cancel{1}\cancel{1}
\end{array}
\Big(321032$$

Thus after *Weigelius*'s way:



DEFINITION XXX.

E*Xtraction of Roots* is a Species of Diviſion, wherein the Quotient is the Root of the given Square or Cube, *&c.* But the Diviſor is not given, neither is it all along the ſame (as it is in Diviſion) but muſt be perpetually found, and they are ſeveral. And as the Squares of Simple Numbers 1, 2, 3, &c. *viz* 1, 4, 9, 16, *&c.* and their Cubes 1, 8, 27, 64, *&c.* may be had immediately out of a Multiplication Table, as alſo their Roots, without any further trouble; and likewiſe in Species, as the Roots of the Square aa or a^2, or of the Cube aaa or a^3, are without doubt (a); ſo if the Square Root be to be extracted out of de, or the Cube Root out of fgm (becauſe the letters are different, and no one can be taken for the Root) the Square Root is commonly noted by this Sign $\sqrt{de}$, the Cube Root by this $\sqrt{C}$, or $\sqrt[3]{fgm}$, &c. as alſo in Numbers that are not perfect Squares (as *e. g.* 2, 3, 5, 6, 7, 8, 10, 11, 15, 17, 19, *&c.*) we can no otherwiſe expreſs the Square Roots, then after this manner $\sqrt{2}$, 7, $\sqrt{19}$, *&c.* and in thoſe that are not perfectly Cubical (as all between 1, 8, 27, 64, *&c.*) we can only expreſs their Cube Roots after ſome ſuch manner, $\sqrt{c.}\,7$, or $\sqrt[3]{7}$. $\sqrt{c.}\,61$, or $\sqrt[3]{61}$ *&c.* Which forms of Roots in ſpecious Computation, we call *Surd Quantities*, in Vulgar Arithmetick *Surd Numbers, i. e.* ſuch as cannot be perfectly expreſſed by any Numbers; altho we have Rules at hand to determine their Values nearer and nearer *ad Infinitum.*

Theſe Rules accommodated to Square and Cube Numbers, *&c.* which otherwiſe are more difficult to be comprehended, appear plain and eaſie to him, who multiplies a Root expreſſed by 2 Letters (called therefore commonly a Binomial) firſt Quadratically, then Cubically, *&c.* For he will have as

CONSECTARYS.

I. THE Square of any aſſumed Root, as alſo, *Prop.* 4. *lib.* 2. *Eucl.* and at the ſame time a general Rule for Extracting the Square Root, all expreſſed in theſe few Notes:

$$aa + 2ab + bb.$$

II. The Cube of the ſame Root, a New Theorem, and at the ſame time a Rule for Extracting any Cube Root, contained in this Theorem:

$$a^3 + 3aab + 3abb + bbb.$$

SCHOLIUM I.

WHich that we may more plainly ſhew, eſpecially as far as it relates to the Rules of Extraction, conſider, 1. That the Root of the Square $aa + 2ab + bb$ is already known (for we aſſumed for the Root the Quantity $a + b$) ſo that now we are to ſee which way this Root is to be obtain'd out of that Square by Diviſion. It will preſently appear, that the firſt Note of the Root a, will come out of the firſt part of the Square aa, and the other part b muſt be obtain'd out of the remainder $2ab + bb$; and ſo as there are 2 Notes of the Root, the Square muſt be diſtinguiſh'd as it were into 2 Claſſes, each of which gives a particular note of the Root. Then it is manifeſt, that the firſt Note of the Root (a) may be obtain'd out of the Square aa by a ſimple Extraction. Now it is evident, if I would have by Diviſion the other Note of the Root, the next following part of the remaining Claſſis muſt be divided by $2a$, the double of the Quotient juſt now found, and that nothing ſhould remain after this Diviſion (for now we have the whole Root $a + b$) you muſt not only ſubtract the Product of the Diviſor and this new Quotient, but alſo the Square of this new Quotient: Which is the Vulgar Method and Rule for the Extraction of Square Roots taught in common Arithmetick.

Likewiſe if you would extract the Root of the above-mentioned Cube, which we already know, having formed it from $a+b$, it is manifeſt, that the firſt Note of the Root a will come out of the firſt part of the Cube a^3, and the other b. muſt be obtain'd out of the remainder $3abb+bbb$, and ſo, as there are two Notes of the Root, the Cube muſt be diſtinguiſh'd, as it were into two Claſſes, each of which will give a particular Note of the Root. Now it is manifeſt, the firſt Note of the Root a is obtained by ſimple Extraction of the Root out of the Cube aaa. It is moreover evident, if I would have by Diviſion the other Note of the Root b, the next remaining part muſt be divided by $3aa$ (the triple Square of the precedent Quotient, or thrice the precedent Quotient multiplied by it ſelf) and, that nothing ſhould remain after this diviſion (for now we have the whole Cube Root $a+b$) you muſt not only ſubtract from the remaining Dividend the Product of the Diviſor, and the new Quotient ($3aab$) but alſo the Product of the Square of the new Quotient, and thrice the precedent Quotient ($3abb$) and moreover the Cube of that new Quotient b^3 : Which is the Method of extracting Cube Roots in Vulgar Arithmetick.

SCHOLIUM II.

FRom what we have ſaid you have alſo the Reaſon of another rule in Arithmetick which teaches how to approach continually nearer and nearer to the Square and Cube Roots of numbers that are not exact Squares and Cubes ; *viz.* by adding to the given Number perpetually new Claſſes and Cyphers or o's, two at a time, to the Square, and three to the Cube, and ſo continue on the operation as before ; which will add Decimal Parts to the Integrals before found ; and the next operation (if you add a ſecond Claſſe of Cyphers) will exhibit Centeſimal Parts, and ſo on *ad Infinitum.* For Example, If I would have the Square Root of 2 pretty near, I can aſſign no nearer whole Number than 1. But by adding a new Claſſe of 2 Cyphers, *i.e.* multiplying the given Number by 100 (whereby the Root is multiplied by 10) you'l have 14, nearly the Root of 200, that is, $\frac{14}{10}$ or $1\frac{4}{10}$ much nearer the Root of 2 than the former ; and thus you may always come nearer and nearer *ad Infinitum,* but never to an exact Root. For if you could have the exact Root of 2 ,or 3, or 5, &c. in any Fraction whatſoever, that Fraction muſt be of ſuch ſort, that its Numerator and Denominator being ſquared, the Fraction thence ariſing muſt exactly equal 2 or 3, or 5, &c. that is, its Numerator muſt be exactly double, or trible, or Quadruple &c. of of the Denominator; which can never be, becauſe both are Squares, and in a Series of Squares no ſuch thing can happen. Hence you have theſe

CONSECTARYS.

III. THat it is a certain mark of Incommenſurability, if one quantity is 1, and other the $\sqrt{2}$, or $\sqrt{3}$, or $\sqrt{5}$, &c.

IV. That theſe ſorts of Quantities are notwithſtanding Commenſurable in their Powers, *i. e.* their Squares are as 1 and 2, or as a number to a number.

V. Thoſe Quantities which are to one another, as 1 and $\sqrt{\sqrt{2}}$, or as $\sqrt{2}$ and the $\sqrt{\sqrt{3}}$ are incommenſurable in Power alſo. Which being rightly underſtood, you may eaſily comprehend ſeveral (*a*) Propoſitions of *lib.* 10 *Eucl.* eſpecially after ſome few things premiſed concerning Reaſon and Proportion.

SCHOLIUM. III.

FRom what we have ſhewn may eaſily be concluded, tha to any propoſed Quantity whatſoever, which *Euclid* call (*b*) *Rational*, and for which we may always put I, there may be ſeveral others both commenſurable and incommenſurable, and that either ſimply or in power ſo; thoſe which are commenſurable to a Rational given Quantity, either Simply or only in Power (which, *e.g.* are to i', as 2, 3, 4, &c. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. or as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{4}}$, &c.) are called alſo *Rational*: but thoſe which are Incom menſurable both ways (*i.e.* both ſimply and in power) as ($\sqrt{\sqrt{2}}$ $\sqrt{\sqrt{\frac{1}{3}}}$, &c.) are called *Irrational.* In like manner the Squar of a given Rational Quantity (as I) is called Rational, an Quantities commenſurable to it (as 2, 3, 4, 5, &c. $\frac{1}{2}$, $\frac{1}{3}$, &c. □) are called alſo *Rational*; but incommenſurable on ($\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{5}}$, &c.) *Irrational*, and the Sides and Roots of the more *Irrational.*

(*a*) *Eucl. l.* 10 *Prop.* 9, 10, 11 12, 13 &c.

(*b*) *lib.* 10. *Def.* 5, 6, 7 8, 9, 10, 11.

DEF

DEFINITION XXXI.

ANy Quantities whatſoever of the ſame kind, whether commenſurable or incommenſurable, equal or unequal, admit of a twofold reſpect or relation of their magnitude, one whereof, when only the difference or exceſs of one above another is reſpected (as 10 which is 3 more than 7) is called an *Arithmetical Reaſon* or *Reſpect*; the other, wherein reſpect is rather had to the Amplitude, whereby one is contained once, or a certain number of times in the other (as 3 is contained thrice in 10 and $\frac{1}{3}$ part more) is called *Geometrical Reaſon*, or by way of Emphaſis, only *Reaſon*, and by others *Proportion*; and this Reaſon or Proportion, if the leſs is exactly contained a certain number of times in the greater (as 3 in 6, or 4 in 12) is generally called, on the part of the greateſt term, *Multiple*, and on part of the leſs, *Submultiple*, and particularly in the firſt Example *double*, when 6 is taken in reſpect to 3, and *ſubduple* when 3 is taken in reſpect to 6; in the other *triple* and *ſubtriple*, &c. If the leſs be contained in the greater once or more times, unity only remaining over and above (as 3 in 4, and 4 in 9) the *Reaſon* or *Proportion* is called *Superparticular* and *Subſuperparticular*, and is noted by the terms *Seſqui* & *Subſeſqui*, joyning the ordinal Name of the leſſer Term; as the the Reaſon of 4 to 3 is called *Seſquitertian*, and contrariwiſe *Subſeſquitertian*; the Reaſon of 9 to 4 is called double *Seſquiquartan*, and contrariwiſe *Subduple Subſequiquartan*, &c. If laſtly, the leſs be contained in the greater once, or a certain number of times, ſeveral units remaining over and above, it is commonly called *Superpartient* Reaſon and is expreſſed by the word *Super* or *Subſuper*, joyned with the adverbial Name of the remaining Parts, and the ordinal Name of the leſſer Term; thus, *e.g.* the Reaſon of 7 to 4, is called *Supertriquartan*, 12 to 5 double Superbiquintan, &c. but when the Quotient ariſes by the diviſion of the greater term by the leſs, and is commonly expreſſed in the ſame words, it is alſo commonly called by the name of *Reaſon* (*e. g.* 2 is the name of the Reaſon of 6 to 3, $2\frac{1}{4}$ of 9 to 4, or contrariwiſe, &c.) as alſo the quotient ariſing by diviſion of the Conſequent by the Antecedent (as $\frac{1}{2}$ in the firſt caſe, $\frac{4}{9}$ in the latter) by which name the antecedent Term of the Reaſon being multiplied, produces its Conſequent; which is evident by naming any Reaſon *e* or *i*, or *o*, &c. Thus if the antecedent Term be called *a* o &c. the Conſequent may be rightly call'd *ea* or *e b*, *oa* or &c. and becauſe in an Arithmetical Relation we only reſpect exceſs of the firſt above the following, or of the following bove the foregoing (which may be called *x* or *z*) if the an cedent (which may be called *a* or *b*) be leſs, the conſequ may properly be called $a+x$ or $b+z$; but if it be greater, t other will be $a-x$ or $b-z$.

CONSECTARYS.

I. WE may hence readily infer, that if the Diameter any Circle be called *a* the Circumference may called *e a*, (for whatever the proportion is between them, may be expreſſed by the Letter *e*) and the Area, according *Conſectary* 2. *Definition* 15, will be $\frac{1}{4}eaa$.

II. If for the Baſe of any Cylinder or Cone you put $\frac{1}{4}ea$ and the Altitude (*b*) the Solidity of that Cylinder may be righ expreſſed by $\frac{1}{4}eaab$, by *Conſect.* 5. *Definit.* 16, and of t Cone by $\frac{1}{12}eaab$, by *Conſect.* 4. *Definit.* 17.

DEFINITION XXXII.

AS the Identity (or ſameneſs) of ſeveral Geometrical Re ſons uſed to be called *Geometrical Proportionality*, or em phatically *Proportion*; ſo the ſimilitude (or likeneſs) of ſever *Arithmetical Reaſons*, is deſervedly call'd *Arithmetical Proportion lity*, or by a particular Name *Progreſſion*; and conſequently tho *Progreſſionals*, or *Arithmetical Proportionals*, which exceed one an ther by the ſame difference, either uninterruptedly or continual as 2, 5, 8, 11, 14, &c. aſcending, or 30, 28, 26, 24, 2 20, &c. deſcending; or interruptedly, as 2 and 5, 7 and 1 11 and 14, &c. aſcending; or 30 and 26, 24 and 20, and 13, &c. deſcending: For which, and all other in what ca ſoever, we may univerſally put this (or ſuch like) continu Progreſſion, *v. g.* a, $a+x$, $a+2x$, $a+3x$, &c. aſcending; a, $a-x$, $a-2x$, $a-3x$, &c. deſcending, but in an interrupt Progreſſion, *v. g.* *b* and $b+z$ and *c* and $c+z$, *c* and $c-z$, *d* a $d-z$, &c. deſcending. Whence you have this

CO

CONSECTARY.

ANY Difference being given, the following Terms of the Progreſſion, continually proceeding from the firſt aſſumed or given one, may be found; as alſo ſeveral Antecedents that interruptedly follow the given or aſſumed ones, *viz.* by adding or ſubtracting the given Difference to or from the former Terms to find the latter.

DEFINITION XXXIII.

IN like manner, ſince *Reaſons* are ſaid to be the ſame, which have the ſame Denomination of Reaſon, thoſe quantities will be proportional which continually aſcend by the ſame denomination of Reaſon, as 2, 4, 8, 16, 32, 64, *&c.* or deſcend, as 81, 27, 9, 3, 1. there by the Denomination of the Reaſon 2, here 3; or that aſcend interruptedly, as 2, 4; 3, 6: 5, 10, &c. or deſcend, as 40, 10; 28, 7; 20, 5; 8, 2, *&c.* Whence you have theſe

CONSECTAYS.

I. HAving two Terms given, or only one with the Denomination of the Reaſon (*e. g.* the Term 2 with the Denomination of the Reaſon 3, or univerſally the firſt Term a with the Denomination of the Reaſon e) it will be eaſie to find as many more Terms of the Geometrical Progreſſion or Proportion as you pleaſe, *viz.* by always multiplying the Antecedent by the Denomination of the Reaſon, that you may have 2, 6, 18, 54, *&c.* or a, $e\,a$, e^2a, e^3a, *&c.* in continued, or 2 and 6, 4 and 12, 5 and 15, *&c.* and $a\,e\,a$, $b\,e\,b$, $d\,e\,d$, *&c.* in diſcontinued or interrupted Proportion.

Thus having rightly underſtood what we have ſaid in this 33 and 31 *Definition,* there will follow theſe *Corollarys* as ſo many Axioms.

II. That equal Quantities have the ſame proportion to the ſame Quantity (*α*) and the ſame has the like to equal Quantities.

III. But a greater quantity has a greater Reaſon to the ſame (*β*) than a leſs, and the ſame has a greater proportion to a leſs Quantity than to a greater.

IV. On the contrary, thoſe that have the (*γ*) ſame proportion to the ſame quantity, and that likewiſe the ſame to them are equal.

V. But that which bears a (*δ*) greater proportion to the ſame is greater; but that to which the ſame bears a greater proportion is leſs.

VI. Proportions equal to one third (*ε*) are alſo equal among themſelves, &c.

DEFINITION XXXIV.

Here remain two things to be taken notice of; Firſt this, If any whole (quantiy) be ſo divided into two equal parts (*α*) that the whole, the greater part and the leſs are in a continual proportion; that (whole) is ſaid to be cut in extreme and mean Reaſon. 2. In a continual Series of that kind of Proportionals (*e. g.* 2. 4. 8. 16. 32, &c. or a, ea, e^2a, e^3a, e^4a, &c.) the Reaſon of the firſt Term to the third (*β*) (2 to 8, or a to e^2a) is particularly called *Duplicate*, and to the 4th (16 or e^3a) *Triplicate*, &c. of that Reaſon which the ſame firſt Term has to its ſecond, or any other antecedent of that Series to its Conſequent: But generally theſe Duplicate and Triplicate Reaſons, &c. as others alſo of the firſt Term to the third or fourth of Proportions continually cohering together, (whether they are the ſame as in the foregoing Examples, or different as in theſe, 2, 4, 6, 18, or a, ea, eia, $eioa$, &c. *viz.* if the name of the firſt Reaſon be e, of the ſecond i, of the third o, &c.) I ſay, the Reaſons of the firſt Term (2 or a) to the third (6 or eia) or to the 4th (18 or $eioa$) are ſaid to be compounded of the continual intermediate Reaſons.

(*α*) *Eucl. lib. Prop.* 7.
(*β*) *prop.* 8.
(*γ*) *prop.* 9.
(*δ*) *prop.* 10
(*ε*) 16.

(*α*) *Eucl. Definit.* 3. *lib.* 6
(*β*) *Eucl. Definit.* 10. *l.* 5.

Now from our general Example, what *Eucli* ſays, is manifeſt,

CONSECTARY I.

THat the denomination of a compounded Reaſon ariſes from the Multiplication of the denominations of the given Simple (a) Reaſons; as the denomination of the reaſon compounded of both (*viz.* a to eia) is produced by multiplying the denomination of the firſt Reaſon e by the denomination of the ſecond Reaſon i, and the denomination of the Reaſon compounded of the three (*viz.* a to $eioa$) is produced by the denomination of the firſt Reaſon e, multiplied by the denomination of the ſecond Reaſon i; and the Product of theſe by the denomination of the third Reaſon o, *&c.*

CONSECTARY II.

SO that it is very eaſie after this way, having never ſo many Reaſons given, whether continued (as 2 to 3, 3 to 6, or a, ea, eia,) or interrupted or diſcrete (as 2 to 3, and 5 to 10, or a to oa, and b to ib) to expreſs their compounded Reaſon: In the firſt caſe it eaſily obtain'd by the bare omiſſion of the intermediate Term or Terms (2 to 6, or a to eia;) and in the other by multiplying firſt of all the Names of the compounding Reaſons among themſelves ($1\frac{1}{2}$ and 2, e. and i.) and by the Product (3 or ei) as the name of the Reaſon compounding the firſt Term (2 or a) that you may have the other 6 or eia) or (if any one had rather do ſo in this latter caſe) by turning the diſcrete or interrupted Reaſons into continued ones, by making as 5 to 10 in the ſecond Reaſon, ſo is the Conſequent of the firſt 3 to 6, or as b to ib, ſo ea to eia,) and then by referring the firſt 2 to the third 6, or the firſt a to the third eia, *&c.* In a word therefore, any Duplicate Reaſon may be appoſitely expreſſed by a to e^2a, and Triplicate by a to e^3a, the one immediately diſcernible by a double, the other by a triple Multiplication into itſelf; as you may alſo commodiouſly, and denote others compounded, *e. g.* of 2 by a to eia, of 3 by a to $eioa$, &c.

(a) *Eucl. l.* 6. *Def.* 5.

SCHOLIUM.

WE will here advertiſe the Reader, that tho the Names o duplicate & triplicate Reaſons,&c. are chiefly appropriate to Geometrical Proportionality, yet the Moderns have alſo accommodated them to Arithmetical alſo; as *e. g.* That *Arithmetica Progreſſion* is called *Duplicate*, whoſe Terms are the Squares o Numbers Arithmetically Proportional (*e. g.* 1, 4, 9, 16, 25 &c.) and *Triplicate*, whoſe Terms are Cubes, (&c. as 1, 8, 27, 64, &c.

DEFINITION XXXV.

AND now at length we may underſtand what Magnitudes Geometers particularly call *like*, or *ſimilar*. Whereas in General one number may be ſaid to be like another, one right Line to another, one obtuſe Angle to another, a Triangle to a Triangle, and the like; but an Acute Angle is not like an Obtuſe one, nor a Triangle like a Parallelogram, or a right Line like a Curve one; or a Square like an Oblong, *&c.* Yet among thoſe Figures which may after that rate in general be ſaid to be like, there is notwithſtanding a great deal of diſſimilitude; therefore in a ſtrict Senſe we call only thoſe Right Lined Figures *ſimilar* or *like* (α) which have each of their Angles reſpectively equal to each of the other (as A and 𝔄) B and 𝔅, C and ℭ, *&c. Fig.* 48.) and the Sides about thoſe equal Angles Proportional, *viz.* as BA to AC, ſo 𝔅𝔄 to 𝔄ℭ, *&c.* (β) and among Solid Figures thoſe are ſaid to be Similar, each of whoſe Planes are reſpectively Similar one to the other, and equal in number on both ſides; as, *e. g.* the Plane AC is ſimilar to the Plane 𝔄ℭ, and CG to ℭ𝔊, *&c.* and ſix in number on both ſides.

Book I.

Section II.

Containing ſeveral Propoſitions demonſtrated from the foregoing Foundations.

CHAP. I.

Of the Compoſition and Diviſion of Quantities.

Propoſition I.

THE Sum and Difference of two unequal Quantities added together, make double of the greateſt.

Demonſtration.

Suppoſe a be the greateſt, b the leaſt, then will their Sum be $a + b$

And their Difference $a - b$

Their Sum $2a$, by *Conſectary* I. *Definition* 27, Q. E. D.

CONSECTARY.

HEnce by a bare Subſumption (α) you have the truth of *Conſect.* 1. *Definit.* 8. that 2 unequal contiguous Angles on the ſame Right Line, ACD and ACE (*Fig.* 49.) *i. e.* if we call the Right Angle BCD or BCE (a) and the difference between the one and the other (b) $a+b$ and $a-b$, make $2a$, *i. e.* are equal to 2 right ones.

(α) *Eucl. lib.* 6. *Definit.* 1.
(β) *lib.* 11. *Def.* 9.
(α) *Eucl.* 13. *l.* 1.

Propoſition II.

IF the Difference of two unequal Quantities be ſubtracted from their Sum, the Remainder will be double of the leaſt.

Demonſtration.

If from the Aggregate or Sum	$a + b$
You ſubtract the Difference	$a - b$
The Remainder will be	$0 + 2b$.

by *Conſectary* 2. *Definition* 27. Q. E. D.

Propoſition III.

BUt if the Sum or Aggregate be ſubtracteed from the Difference, the remainder is ſo much leſs than nothing, as is the double of the laſt Quantity.

Demonſtration.

For if from the Difference	$a - b$
You ſubtract the Sum or Aggregate	$a + b$
The Remainder will be ——	$0 - 2b$,

by *Conſectary* 3 of the aforeſaid *Definition.* Q. E. D.

Propoſition IV.

IF a Poſitive Quantity be multiplied by a Negative one, or contrariwiſe, the Product will be a Negative Quantity.

Expoſition.

If $a-b$ *is to be multiplied by* c; *it is certain, that* a *multiplied by* c, *makes* ac *a Poſitive Quantity, by* Conſect. 1. Definit. 28. *Moreover* b *by the ſame* c (*a Negative by a Poſitive*) *will make* $-bc$; *and ſo the whole Product of* $a-b$ *by* $+c$, *will be* $ac-bc$.

Demon-

Demonſtration.

Suppoſe $a-b=e$; therefore ec will be = to the Product of $a-b$ by c: and ſince $a-b$ is $=e$ by the Hypoth. adding on both ſides b, you'l have $a=e+b$. by *Schol. Definit.* 26. and multiplying both ſides by c, $ac=ec+bc$, by *Conſect.* 2. *Definit.* 28. and by further ſubtracting from each ſide bc, you'l have $ac-bc=ec$, that is, to the Product of $a-b$ by c. Q. E D.

CONSECTARY.

Since $ac-bc$ is the Product of $a-b$ by c, it is manifeſt alſo, that if $ac-bc$ be divided by c, you'l have $a-b$ for the Quotient; and ſo always a Poſitive Quantity (as ac) divided by a Poſitive one, c, will give a Poſitive Quotient; but a Negative Quantity $-bc$ divided by a Poſitive one, will give a Negative Quotient.

Propoſition V.

If a Negative Quantity be multiplied by a Negative one, the Product will be Poſitive.

Expoſition.

Suppoſe $a-b$ is to be multiplied by $-c$; it is certain, that a multiplied by $-c$ will give the Negative Quantity $-ac$, by *Prop.* 4. but $-b$ multiplied by the ſame $-c$ will produce $+bc$, and ſo the whole Product will be $-ac+bc$.

A Demonſtration like the former.

Suppoſe $a-b=e$, then will $-ec=$ the Product of $a-b$ by $-c$: and ſince $a-b$ is $=e$, adding b on both ſides you'l have $a=e+b$, by *Schol. Definit.* 26. and multiplying both ſides by $-c$, you'l have $ac=-ec-bc$, by *Prop.* 4. and *Conſect.* 2. *Definit.* 28. and by adding bc on both ſides, you'l have

$-ac+bc=-ec$, i. e. to the Product of $a-b$ by $-c$ Q. E. D.

CONSECTARYS.

I. Since therefore $a-c+bc$ ariſes from $a-b$ by $-c$, it is manifeſt, that if $-ac+bc$ be divided again by $-c$, you will again have $a-b$, and conſequently a Negative Quantity divided by Negative, will give a Poſitive Quotient, but a Poſitive Quantity $+bc$ divided by a Negative one, will give a negative Quotient $-b$.

II. We have therefore the Foundation and Demonſtration of the Rules of Specious Computation, in the multiplication and diviſion of Compounded Quantities, *viz.* that the ſame Signs multiplied together (as + by + or — by —) give +; but different (as + by — or — by +) give the Sign —: Which Rules the following Examples will Illuſtrate, as alſo ſeveral other we ſhall meet with in the following Chapter.

Multiplication.

$$\begin{array}{l} a+b \\ a-b \\ \hline aa+ab \\ \quad\;\; -ab-bb \\ \hline aa-bb \end{array} \qquad\qquad \begin{array}{l} a-b \\ c-d \\ \hline ac-cb \\ \quad\;\; -ad+bd \\ \hline ac-cb-ad+bd \end{array}$$

$$\begin{array}{l} 3d-4e+2f \\ 5d-g \\ \hline 15dd-20de-10df \\ \qquad\; -3dg+4ge-2gf \\ \hline 15dd-20de+10df-3dg+4ge-2gf. \end{array}$$

Diviſion.

by $\begin{array}{l} aa-bb \\ a+b \end{array}$ $($ $a-b$ | by $\begin{array}{l} ac-ad-bc+bd \\ e+\;\;d\quad\;\; e-\;\;d \end{array}$ $($ $a-b$

CHAP. II.

CAAP. II.

Of the Powers of QUANTITIES.

Containing (after a compendious Way) moſt part of the 2d Book of *Euclid*; and the Appendix of *Clavius* to *Lib.* 9. *Prop.* 14.

Propoſition VI.

IF any whole Quantity be divided into two parts (α) the Rectangle contained under the whole, and one of its parts, is equal to the Square of the ſame part, and the Rectangle contain'd under both the parts.

(a) *Eucl. lib.* 2. *prop.* 3. *A. C. alſo the third.*

Demonſtration.

Let $a+b$ repreſent the whole $\qquad a+b$

b one part of it, or $\qquad a$ the other.

$ab+bb$ the Rectangle, $\qquad aa+ab$ the Rectangle.

(See *Fig.* 50.) Q. E. D.

Propoſition VII.

IF a whole Quantity be divided into two parts (β) the Square of the whole is equal to the Squares of both thoſe parts and 2 Rectangles contained under them.

(β) *Eucl. Prop.* 4. *lib.* 2.

Demonſtration.

This is evident from the preceding, and may moreover thus appear further,

Let the Parts be *a* and *b*, then will the whole be $a+b$
Which if you multiply by it ſelf $a+b$

$$\begin{array}{r} aa+ab \\ ab+bb \\ \hline aa+2ab+bb \end{array}$$

You have the Square $aa+2ab+bb$
(See *Fig.* 51. N. 1.) Q. E. D.

CONSECTARYS.

I. HEnce you have the Original Rule for Extracting of Square Roots, as we have ſhewn after *Definition* 30. and here have further Illuſtrated in *Scheme* N° 2.

II. Hence it naturally follows, that the Square of double any Side is Quadruple of the Square of that Side taken ſingly.

III. Hence alſo you have the addition of ſurd Numbers, or in general of ſurd Quantities, by help of the following Rule (ſuppoſing in the mean while their Multiplication:) Suppoſe theſe 2 Surds $\sqrt{8}$ and $\sqrt{18}$, or more generally $\sqrt{75aa}$ and $\sqrt{27aa}$, are to be added together; firſt add their Squares 8 and 18, &c. then double their Rectangle ($\sqrt{144}$) that is, multiply it by the $\sqrt{4}$, and then the double of this $\sqrt{576}$, *i. e.* having extracted the Square Root, (24) and added it to the Sum of the firſt Squares (26) the Root of the whole Summ (50) *viz.* $\sqrt{50}$, is the Sum of the two ſurd Quantities firſt propoſed.

SCHOLIUM.

BUT if it happens that the Root of the double Product cannot be expreſſed by a Rational Number (as, when the propoſed Quantities are Surds, as $\sqrt{3}$ and $\sqrt{7}$, to whoſe Squares $3+7$, *i. e.* 10, you muſt add the double Product of $\sqrt{7}$ by $\sqrt{3}$, *i. e.* $\sqrt{84}$, which cannot be expreſſed by a Rational Number) then that double Product muſt be joined under a Surd Form, or Radical Sign, to the Sum of the Squares (thus, *viz.* $10+\sqrt{84}$) and to this whole Aggregate prefix another Radical Sign, thus, $\sqrt{10+\sqrt{84}}$; or alſo you may only ſimply join the Surd Quantities propoſed by the Sign $+$ thus, $\sqrt{3}+\sqrt{7}$. Here alſo you may note, that the two *Surd Quantities* propoſed in the firſt caſe of *Conſectary* 2. are called *Communicants*; in the other caſe of this *Scholium*, *Non-Communicants*: For in this caſe each quantity under the Radical Sign may be divided by ſome Square, and have the ſame Quotient (*e. g.* 8 and 18, may be divided the firſt by 4, the other by 9, and the Quotient of both will be 2; likewiſe $75aa$ and $27aa$ may be divided, the one by $25aa$, the other by $9aa$, the Quotient of both being 3; and then if the Quotient on both ſides be left under the Radical Sign, and the Root of the dividing Square ſet before it, the ſame quantities will be rightly expreſſed under this form: $2\sqrt{2}$ and $3\sqrt{2}$, alſo $5a\sqrt{3}$ and $3a\sqrt{3}$; and then the addition is eaſie, *viz.* only collecting or adding together the Quantities prefixt to the Radical Sign; ſo that the Sums will be of the one $5\sqrt{2}$ and of the other $8a\sqrt{3}$, which are indeed the ſame we have ſhewn in *Conſect.* 2. For if contrarywiſe we ſquare the Quantities that ſtand without, or are prefixt to the Radical Sign, and then ſet thoſe Squares (25 and $64aa$) under the Radical Sign, multiplying by the Number prefixt to it, you'l have for the one $\sqrt{50}$, for the other $\sqrt{192aa}$ (*Conſect.* after *Schol- Prop.* 22.)

Propoſition VIII.

IF any whole Quantity (*viz.* Line or Number) be divided (*a*) into two equal parts, and two unequal ones, the Rectangle of the unequal ones, together with the Square of (the intermediate part or) the difference of the equal part from the unequal one, is equal the Square of the half.

(*a*) *Eucl.* & *Clav.* 5.

An Univerſal **Demonſtration.**

Suppoſe the parts to be a and a, and the whole $2a$; let one of the unequal Parts be b, the other will be $2a-b$, and the difference between the equal and unequal part $a-b$.

$$\begin{array}{lr} \text{The equal ones} & 2a-b \\ & +b \\ \hline \text{Rectangle} & 2ab-bb \end{array} \qquad \begin{array}{lr} \text{Difference} & a-b \\ & a-b \\ \hline & aa-ab \\ & -ab+bb \\ \hline & aa-2ab+bb\,\square \end{array}$$

The *S*um will be aa (the other parts deſtroying one another) Q. E. D. (*Vid. Fig.* 52.)

Propoſition IX.

IF to any whole Quantity divided into two equal parts (*a*) you add another Quantity of the ſame kind, the Rectangle or Product made of the whole and the part added, multiplied by that part added, together with the ſquare of the half, will be equal to the Square of the Quantity compounded of that half, and the part added.

(*a*) *Eucl.* & *Clav.* 6.

Demonſtration.

Let the whole be called $2a$, the part added b, then the quantity compounded of the whole and the part added will be $2a+b$; and that compounded of the half and the part added $a+b$.

The Quantity compounded of the whole, and the part added,

$$\begin{array}{lrrr} \text{is,} & 2a+b & \text{the half } a & \text{Comp. } a+b \\ \text{Multip. by the part added} & b & a & a+b \\ \hline & & & aa+ab \\ & & & ab+bb \\ \hline & 2ab+bb\,\square & \text{of the half } aa = & \square\, aa+2ab+bb \end{array}$$

(*Vid. Fig.* 53.) Q. E. D.

Propoſitiou X.

IF a Quantity be divided any how into (*b*) two parts, the Square of the whole, together with that of one of its parts, is equal to two Rectangles contained under the whole and the firſt part, together with the *S*quare of the other part.

(*b*) *Eucl.* & *Clav.prop.*7.*l.*2

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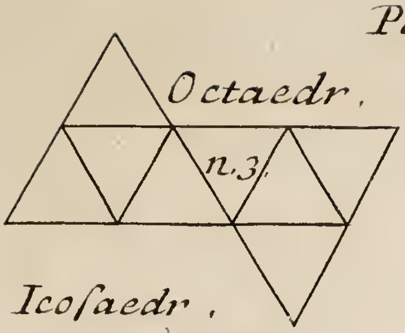
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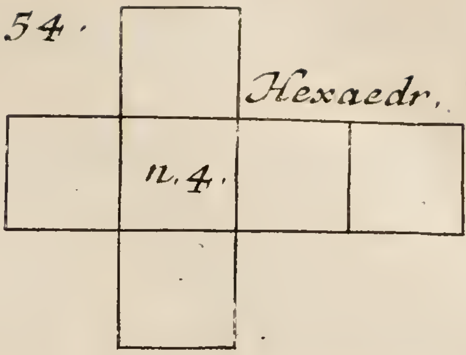
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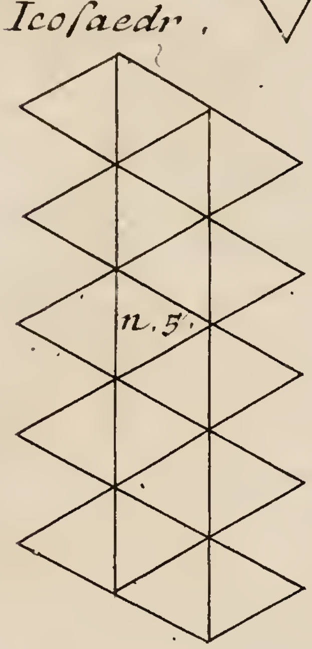
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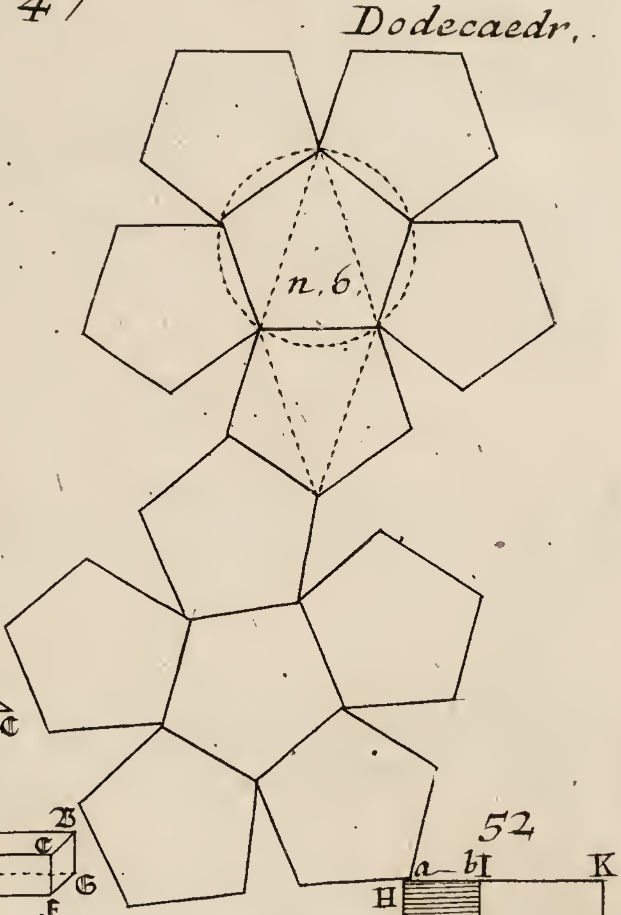
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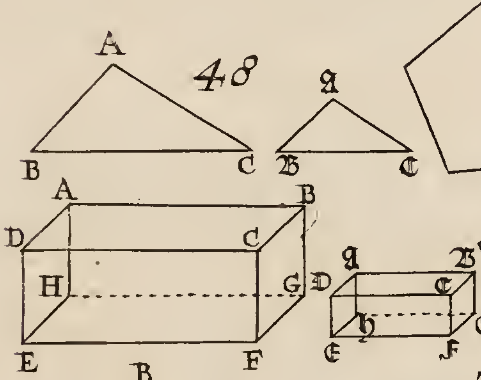
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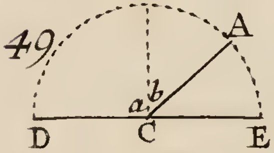
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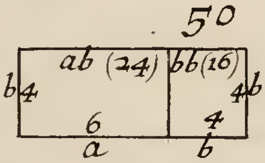
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a
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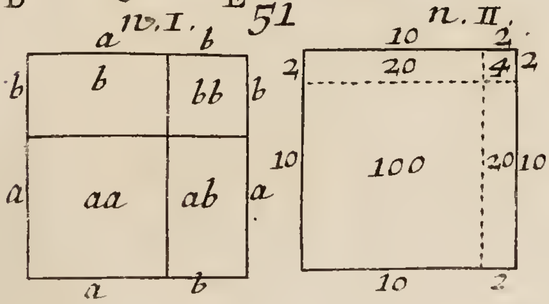
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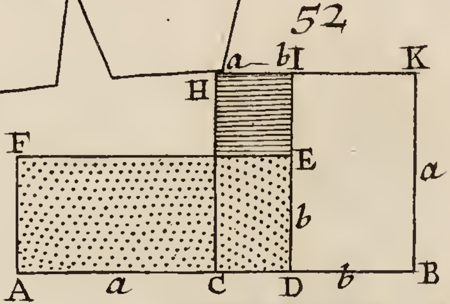
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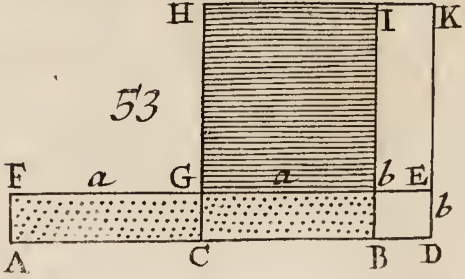
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The Univerſal **Demonſtration.**

Let a be one part and b the other, the whole	$a + b$
$a + b$ the whole.	The whole $a + b$
a the firſt part *	$a + b$
$aa + ab$	$aa + ab$
2	$ab + bb$
$2aa + 2ab$ the double rectangle	□ of the whole $aa + 2ab + bb$
add bb the □ of the other part	* add aa
Sum $2aa + 2ab + bb$	$=$ to the Sum . . . . $2aa + 2ab + bb$
(*Vid. Fig.* 54. N° 1.)	Q.E.D.

CONSECTARY.

Hence you have the Subtraction of Surd Numbers, or more generally of Surd Quantities, by help of the following Rule.

Add the Squares of the given Roots according to Conſect. 3. Prop. 7. *and from their Sum ſubtract the double Rectangle of their Roots; the Root of the Remainder will be the difference ſought of the given Quantities.*

As, if the $\sqrt{8}$ (BC) is to be ſubtracted from $\sqrt{50}$ AC (*Fig.* 54, N° 1.) you muſt add 50, (*i. e.* the whole Square AD) and 8, (*i. e.* the other Square ſuperadded DE,) and the Sum will be 58, equal to the two Rectangles AF and FE + □ GH, by this Prop. I find therefore thoſe two Rectangles by multiplying $\sqrt{50}$ by $\sqrt{8}$, and then the Product $\sqrt{400}$ by 2 or $\sqrt{4}$, thereby to obtain the double Rectangle $\sqrt{1600}$, *i. e.* (having actually Extracted the Root) 40. This double Rectangle therefore or 40, being ſubtracted out of the ſuperiour Sum, the remainder 18 will be the □GH, and ſo its Root (*viz.* $\sqrt{18}$) gives the required Difference between the given Surd Quantities.

SCHOLION.

But this Subtraction may be performed yet a ſhorter way, if each quantity under its Radical can be divided by ſome ſquare,

ſquare ,ſo that the ſame Quotient may come out on both ſides ; that is, if the Surd Quantities are *Communicants*, as *e. g.* $\sqrt{50}$ (the number 50 being divided by 25) is equal to $5\sqrt{2}$ and $\sqrt{8}$ to $2\sqrt{2}$; for then the numbers prefixt to the Radical Sign being ſubtracted from one another (*viz.* $2\sqrt{2}$ from $5\sqrt{2}$) you'l have immediately the remainder or difference $3\sqrt{2}$, *i. e.* $\sqrt{18}$. But if the propoſed Quantities are not Communicants (as if the $\sqrt{3}$ is to be ſubtracted from $\sqrt{7}$) the remainder may be briefly expreſſed by means of the Sign — thus, $\sqrt{7}-\sqrt{3}$, or according to the foregoing *Conſectary*, thus, $\sqrt{10-\sqrt{84}}$.

Propoſition XI.

IF *any Quantity be divided into two parts,* (a) *the Quadruple Rectangle contained under the whole and one of its parts, together with the Square of its other parts, will be equal to the Square of the Quantity compounded of the whole and the other part.*

Demonſtration.

Suppoſe $a+b$ the whole.
b one part.

$ab+bb$ the Rectangle of theſe two.
mult. by 4

$4ab+4bb$ the Quadruple Rectangle.
Add aa the Square of the other part.

Sum $aa+4ab+bb$

The Quantity compounded of the whole and the firſt part $\left.\right\}$ $a+2b$
$a+2b$

$aa+2ba$
$2ba+4bb$

$aa+4ba+4bb=$
Square of the Compound Quantity Q. E.D.
(*Vid. Fig.* 55.)

(a) *Eucl. & Clav. prop.* 8.

Propoſition

Propoſition XII.

IF any Quantity be divided into two equal parts (β) and into two other unequal ones, the Squares of the unequal parts taken together will be double the Square of half the quantity, and the Square of the difference, *viz.* of the equal and unequal part, taken together.

(β) *Eucl* & *Clav. prop.*9.

Demonſtration.

Suppoſe the equal parts to be *a* and *a*, the difference (*b*) the greater of the unequal Parts to be $a+b$, the leſs $a-b$.

The greater part $a+b$	The leſs $a-b$	Half	Difference
$a+b$	$a-b$	a	b
		a	b
$\square aa+2ab+bb$	$aa-2ab+bb$	$\square aa$	$\square bb$
Sum of theſe $2aa+2abb$		Sum $aa+bb$	

Q. E. D. (*Vid. Fig.* 56.)

Propoſition XIII.

IF to any whole Quantity (α) divided into two equal parts there be added another Quantity of the ſame kind, the Square of the Quantity compounded of the whole, and the quantity added, together with the ſquare of the quantity added, will be double the ſquare of the half the quantity, and the ſquare of the Sum of the half and the part added taken together.

(α) *Eucl.* & *Clav.* x.

Demonſtration.

Suppoſe the whole to be 2 *a*, the half parts *a* and *a*, the quantity added *b*; then the quantity compounded of the whole and the quantity added, will be $2a+b$, and that of the half and the quantity added $a+b$.

Comp. of the whole and quantity added $\left.\begin{matrix}2a+b\\2a+b\end{matrix}\right.$

$4aa+2ab$

$2ab+bb$

☐ $4aa+4ab+bb$

☐ Qu. added, bb

Sum $4aa+4ab+2bb$

Half a

a

☐ aa

Qu. compouded of hal
and qu. added, $a+$

$a+$

☐$aa+2ab+$

Sum $2aa+2ab+$

Manifeſtly the half of the for
mer Sum. Q.E.D.

CHAP. III.

Of Progreſſion, or Arithmetical Proportionals.

Propoſition XIV.

IF there are 3 Quantities in continued Progreſſion, or Arithmetical continued Proportion, the Sum of the Extrem is double of the middle Term.

Demonſtration.

Such are *e.g.* a, $a+x$, $a+2x$ aſcending,
or a, $a-x$, $a-2x$ deſcending.
By *Definition* 32. the Sum of the Extremes in the firſt $2a+2x$, in the latter $2a-2x$; in both manifeſtly double the middle Term Q.E.D.

Propoſition XV.

IF there are 4 of theſe Continued Proportionals, the Sum the Extreme Terms is equal to the Sum of the me Terms.

Demonſtration.

Such are *e. g.* a, $a+x$, $a+2x$, $a+3x$, &c. aſcending, or a, $a-x$, $a-2x$, $a-3x$,&c deſcending; in the one the Sum of the Extremes is $2a+3x$, in the other $2a-3x$; and alſo of the means $2a+3x$ and $2a-3x$ Q. E. D.

Propoſition XVI.

IF there are never ſo many of theſe continued Proportionals, the Sum of the Extremes is always equal to the Sum of any 2 other of them, equally remote from the Extremes, or alſo double of the middle Term, if the number of the Terms is odd.

Demonſtration.

Suppoſe a, $a+x$, $a+2x$, $a+3x$, $a+4x$, $a+5x$, $a+6x$, *&c.* or a, $a-x$, $a-2x$, $a-3x$, $a-4x$, $a-5x$, $a-6x$, and the Sum of the Extremes, as alſo of any 2 equally remote from the Extremes, and the double of the middle Term is in the firſt Series $2a+6x$, in the latter $2a-6x$, *&c.* Q.E.D.

SCHOLIUM I.

NOR can we doubt but that this will always be ſo, how far ſoever the Progreſſion be continued; if you conſider that the laſt Term contains in it ſelf the firſt, and moreover the difference ſo many times taken, as is the number of Terms excepting one, but that the firſt has no difference added to it; and therefore tho the laſt ſince one contains one difference leſs than the laſt; the ſecond on the contrary has one more than the firſt, and conſequently the Sum of the one will neceſſarily be equal to the Sum of the other; and in like manner the laſt except two, contains two Differences leſs than the laſt; but, on the contrary, the third exceeds the firſt by a double Difference, the double Difference being added to it, *&c.* as is obvious to the Eye in our firſt univerſal Example. Hence you have theſe

CONSECTARYS.

I. YOU may obtain the Sum of any Terms in Arithmetical Proportion, if the Sum of the Extremes be multiplied by half the number of Terms, or (which is the ſame thing) half the *S*um by the number of Terms,

II. To obtain therefore the Sum of 600, or never ſo many ſuch Terms, you need only have the Extremes and the number of Terms: So that you have a very compendious Way of proceeding in Queſtions that are ſolvible by theſe Progreſſions, if, having the firſt Term and Difference of the Progreſſion given, you can obtain the laſt, neglecting the intermediate ones.

III. But you may obtain the laſt Term, by Multiplying the given given Difference by the given Number of Terms leſſened by Unity, and then adding the firſt Term to the Product; as is evident from the preceding *Scholium.*

IV. Hence we may eaſily deduce this Theorem, that the Sum of any Arithmetical Progreſſion beginning from o, is ſubduple of the Sum of ſo many Terms, equal to the greateſt, as is the number of Terms of that Progreſſion. For if the firſt Term is o and the laſt x, and the given number of Terms a, the *S*um of the Progreſſion will be $\frac{1}{2} a x$, by *Conſect.* 1. but the Sum of ſo many Terms equal to the greateſt, ax. Q.E.D.

SCHOLIUM II.

NOw if any one would be ſatisfied of the truth of this laſt *Conſect.* without the literal or ſpecious Notes, let him conſider, that if the firſt Term be ſuppoſed to be o, the laſt (whatever it is) will be the ſum of the Extremes. The laſt therefore multiplied by half the number of Terms, gives the Sum of the Progreſſion, by *Conſect.* 1. and the ſame laſt Term multiplied by the whole number of Terms, gives the Sum of ſo many Terms equal to the greateſt. But that this muſt needs be double of the precedent 'tis evident, becauſe any Multiplicand being multiplied by a double multiplier, muſt needs give a double Product. Now as this Conſectary will be of ſingular Uſe to us hereafter for Demonſtrating ſeveral Propoſitions, ſo the three former are the very ſame Practical Rules of Arithmetick, which are commonly made uſe of in Arithmetical Progreſſions; for the Illuſtration whereof *Swenterus* gives us ſeveral Ingenious Examples in his *Delic.* part 1. Queſt. 70. *&c.*

CHAP. IV.

Of Geometrical Proportion in General.

Propoſition XVII.

IF there are three Quantities continually (*a*) Proportional, the Rectangle of the Extremes, is equal to the Square of the mean Term.

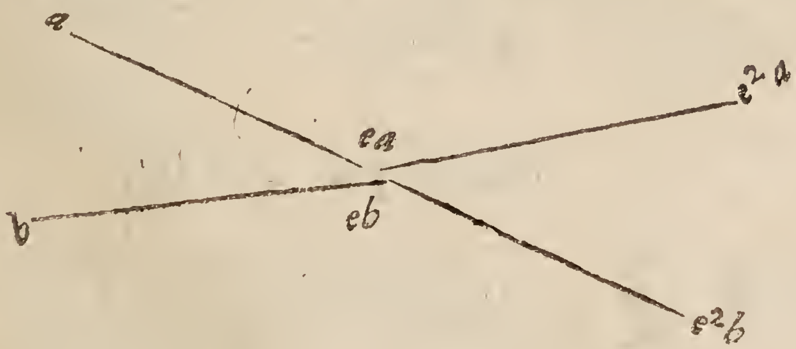
Demonſtration.

Such are *e. g.* $a,\ ea,\ e^2a,$
The Extremes e^2a
a
Rectangle e^2a^2

The mean Term, ea
ea
Square e^2a^2 Q.E.D.

SCHOLIUM.

MOreover if three Quantities on each ſide are in the ſame Continual Proportion, as



the Rectangles of the Extremes made Croſs-ways, are equal to the Rectangle of the mean Term; being every way e^2ab.

Whence by the way may appear that Propoſition of *Archimedes*: (β) *That the Surface of a Right Cone is equal to the Circle, whoſe Radius is a mean Proportional between the Side of that Cone and the Semidiameter of the Baſe.* For ſuppoſe EF to be a mean Proportional between the ſide of the Cone BC (*Fig.* 57.) and the Semidiameter of the Baſe CD, ſince an equal number of Peripherys anſwer to an equal number of Radii in the ſame Proportion; half the Product of the firſt Line BC into the laſt Periphery, $\frac{1}{2}e^2ab$ (that is, by *Conſect.* 4. *Definit.* 18. the Surface of the given Cone) will be equal to half Product of the mean Line into the mean Periphery, $\frac{1}{2}e^2ab$ (*i. e.* by *Conſect.* 2. *Definit.* 15.) to the Area of the Circle of the mean Proportional EF. Q. E. D.

(α) *Eucl.* 17 *l.* 6 & 20 *l.* 7.
(β) *lib.* 1. *de Sphær.* & *Cyl. Prop.* 14.

The ſame Propoſition of *Archimedes* may alſo be Demonſtrated after this Way: If the ſide of the Cone BC be called b, and the Semidiameter of the Baſe a)ſo that the Periphery may, by *Conſect.* 1. *Definit.* 31. be $2ea$, and ſo the Surface of the Cone, by *Conſect.* 4. *Definit.* 18. eab) the $\sqrt{ab}$ will be a mean proportional between b and a, by this 17th *Propoſition*; which being taken for *Radius*, the whole Diameter will be $\sqrt{2ab}$, and the Periphery $2e\sqrt{ab}$; therefore by *Conſect.* 2. *Definit.* 15. half the Radius $\frac{1}{2}\sqrt{ab}$ multiplied by the Periphery (ſince $\sqrt{ab}$ multiplied by $\sqrt{ab}$ neceſſarily produces ab) will give you the Area of the Circle by that mean (*a*) Proportional, equal to the Surface of the given Cone, which before was expreſſed in the ſame Terms. Q. E. D.

Hence alſo naturally flows this other Propoſition, That the Surface of the Cone ($\frac{1}{2}e^2ab$) is to its Baſe ($\frac{1}{2}ab$) as the Side of the Cone (e^2b) is to the Radius of the Baſe b, as may appear from the Terms.

Propoſition XVIII.

IF (*b*) 4 Quantities are Proportional, either continuedly or diſcretely, the Product of the Extremes is equal to the Product of the Means.

(*a*) *lib.* 1. *de Sphær.* & *Cyl. prop.* 15.
(*b*) *Eucl.* 16 *l.* 6 & 19. *l.* 7.

Demon-

Demonſtration.

Suppoſe one Continual Proportional, a, ea, e^2a, e^3a.

Extremes e^3a | Means e^2a
a | ea

Prod. e^3aa = Prod. e^3aa. Q.E.D.

SCHOLIUM.

ON this Theorem is founded the *Rule of Three* in Arithmetick; ſo called becauſe having 3 Numbers, (2. 5. 8.) it finds an unknown fourth Proportional. For altho this fourth be, as we have ſaid, unknown, yets its Product by 2 is known, becauſe the ſame with the Product of the Means, 5 and 8. Wherefore the Rule directs to multiply the third by the ſecond, that you may thereby obtain the Product of the Extremes: which divided by one of the Extremes, *viz.* the firſt, neceſſarily gives the other, *i. e.* the fourth ſought.

Propoſition XIX.

IF 2 Products (on the other ſide) ariſing from the Multiplication of 2 Quantities, are equal, thoſe 4 Quantities will be at leaſt directly Proportional.

Demonſtration.

Suppoſe eba be the equal Product of the Extremes, and eab of the Means; the Extremes will either be eb and a, or e and ba, or b and ea, as alſo the Means. But what way ſoever either is taken, there can be no other Diſpoſition or placing of them, than one of the following.

1 eb eb a a
——— ee ab
——— ea b; or inverſly.
——— a eb
——— ab e

——— *b* *ea*
2 *e* *e* *ba* *ba*
——— *eb* *a*
——— *ea* *b*; or inverſly.
——— *ba* *e*
——— *a* *eb*
——— *b* *ea*
3 *b* *b* *ea* *ea*
——— *a* *eb*
——— *ba* *e*; or inverſly.
——— *ea* *b*
——— *eb* *a*
——— *e* *ba*; or inverting the Order of them all.

In all theſe Diſpoſitions there may be immediately ſeen a Geometrical Proportion, by what we have in *Definition* 31 and 33.

CONSECTARSY.

I. As we have ſhewn one *S*ign of Proportionality in the Definition of it, *viz*. That the ſame Quotient will ariſe by dividing the Conſequents by the Antecedents; ſo now we have another *S*ign of it, *viz*. The Equality of the Products of the Extremes and Means.

II. By a bare *S*ubſumption may hence appear the Truth of *Prop*. 14. *lib*. 6. *Euclid*. at leaſt partly: Which we ſhall yet more commodiouſly ſhew hereafter.

Propoſition XX.

If there are never ſo many Continual Proportionals, the Product of the Extremes is equal to the Product of any 2 of the Means that are equally diſtant from the Extremes, as alſo to the *S*quare of the mean or middle Term, if the Terms are odd.

Demonſtration.

*S*uch are *e. g.* a, ea, e^2a, e^3a, e^4a, e^5a, e^6a, &c. and the Product of the Extremes, and of any two Terms equally remote from them, and the Square of the mean or middle Term, every where e^6aa. Q. E D.

SCHOLIUM I.

NOR can there be any doubt but this will always be ſo, how far ſoever the Progreſſion is continued; if you conſider that the laſt Term always contains the firſt, by way of Reaſon, ſo many times multiplied as is the place of that Term in the rank of Terms, excepting one. Altho therefore the laſt Term but one is in one degree of its Reaſon leſs than the laſt, the ſecond on the contrary, is in one more than the firſt; therefore the Product of the one will neceſſarily be equal to the Product of the other. Thus alſo the laſt Term but two is 2 Degrees of Proportion lower than the laſt; but then the third being to be multiplied into that, exceeds the firſt by 2 Degrees of the Proportion, *&c.* as may be ſeen in our Univerſal Example. Hence you have the following

CONSECTARIES.

I. HAving ſome of the Terms given in a Continual Proportion (*e.g.* ſuppoſe 10) you may eaſily find any other that ſhall be required (*e.g.* the 17th) as the laſt; If the 2 Terms given, being equally remote from the firſt and that required (as are *e.g.* the eigth and tenth) be multiplied by one another, and this Product, like that alſo of the Extremes, be divided by the firſt.

II. But this may be performed eaſier, if you moreover take in this Obſervation, That if, *e.g.* never ſo many places of proportionals, paſſing over the the firſt, be noted or marked by Ordinals or Numbers according to their places (as in this univerſal Example)

$$a,\ ea,\ e^2a,\ e^3a,\ e^4a,\ e^5a,\ e^6a,$$

I. II. III. IV. V. VI.

The place of the 7th Term is (*e.g.*) VI. (and ſo the place of any other of them being leſs by Unity than its number is among the Terms) and alſo compoſed of the places of any other equally diſtant from the Extremes, *e. g.* V. and I. IV. and II. or twice III. *&c.*

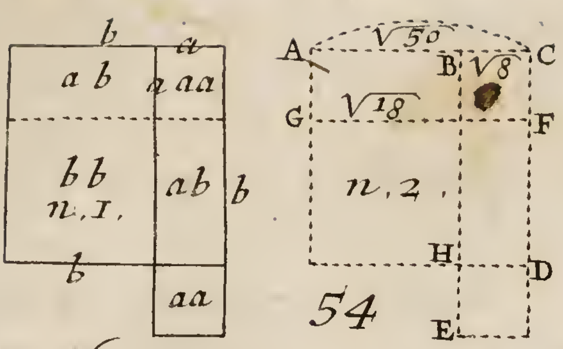
III. Here you have the Foundation of the Logarithms, *i. e.* of a Compendious Way of Arithmetick, never enough to be praiſed. For if, *e. g.* a rank of Numbers from Unity, continually Proportional, be ſigned or noted with their Ordinals, as we have ſaid, as Logarithms,

1. 2. 4. 8. 16. 32. 64. 128. 256, &c.
I. II. III. IV. V. VI. VII. VIII.

and any two of them (as 8 and 32) are to be multiplied together; add their Logarithms III and V, and their Sum VIII, gives you the Logarithm of their Product 256, as the Term equally remote from the 2 given ones and the firſt, and ſo whoſe Product with the firſt (which is Unity) *i. e.* it ſelf will be equal to the Product of the Numbers to be multiplied: And contrariwiſe, if, *e. g.* 128 is to be divided by 4, ſubtracting the Logarithm of the firſt II from the Logarithm of the ſecond VII, the remaining Logarithm V points out the number ſought 32; ſo that after this way the Multiplication of Proportionals is, by a wonderful Compendium, turned into Addition, and their Diviſion into Subtraction, and Extraction of the Square Root into Biſecting or Halving, (for the Logarithm of the Square Number 16 being Biſected, the half II gives the Root ſought 4) of the Cube Root into Triſection (for the Logarithm of the Cube 64 being Triſected, the third part gives the Cubick Root ſought 4).

SCHOLIUM II.

THat we may exhibit the whole Reaſon of this admirable Artifice (which about 35 years ago was found out by the Honourable Lord *John Naper* Baron of *Merchiſton* in *Scotland*, and publiſhed ſomething difficult, but afterwards render'd much eaſier and brought to perfection by *Henry Briggs*, the firſt *Savilian* Profeſſor of Geometry at *Oxford.*) I ſay that we may exhibit the whole Reaſon of it in a Synopſis, after an eaſie way; when its uſe appear'd ſo very Conſiderable in the great Numbers in the Tables of Sines and Tangents, nor yet could they be uſeful without mixing vulgar Numbers with them, eſpecially in the Practical Parts of Geometry, the buſineſs was to ac-


b
a
a b
a
aa
b b
n.1.
ab
b
b
aa
A
√50
C
B
√8
G
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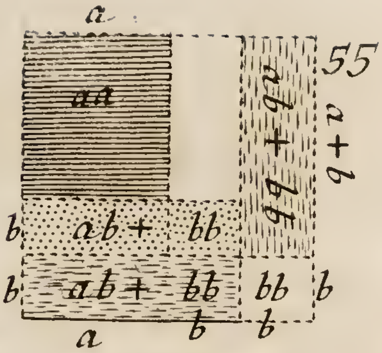
F
n.2.
H
D
E
54


a
55
aa
ab + bb
a + b
b
ab + bb
b
ab + bb
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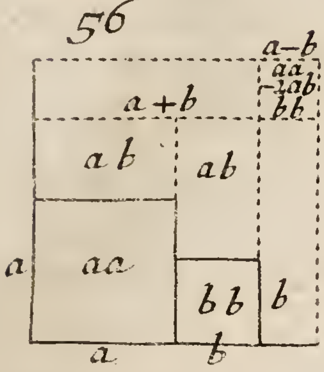
b
a
b
b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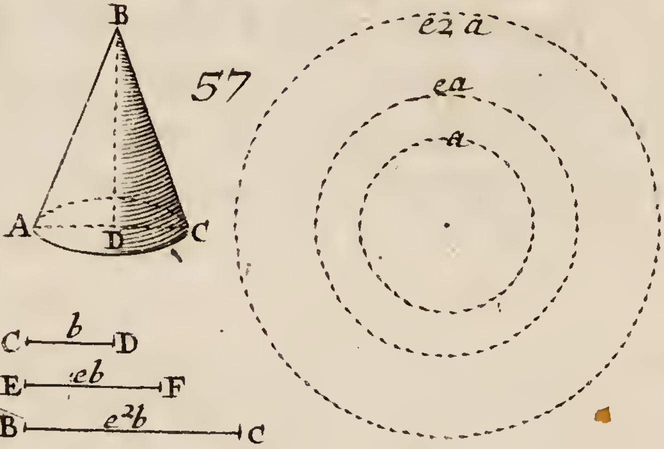
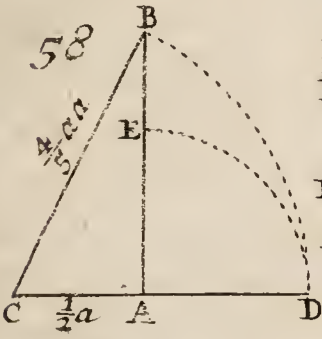
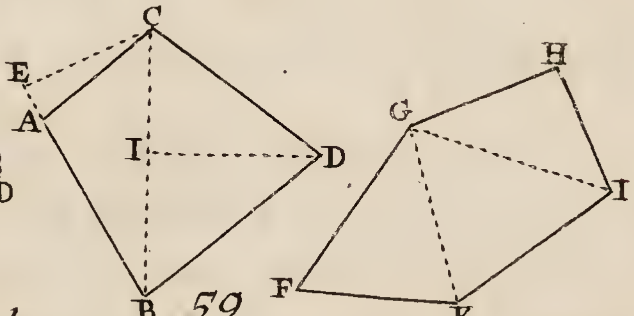
a−b
aa
−2ab
bb
a + b
a b
ab
a
aa
b b
b
a
b


B
57
A
D
C
C
b
D
E
eb
F
B
e²b
C
e2 a
ea
a


58
B
4/5 aa
E
C
½a
A
D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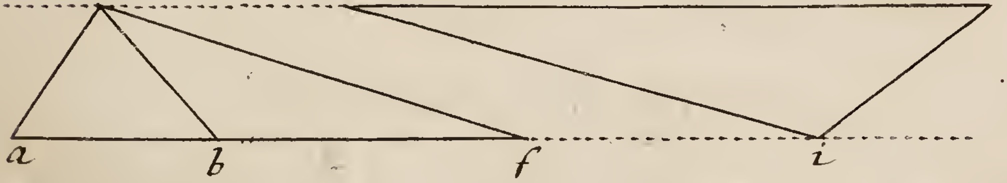
B
59
H
G
I
F
K


c
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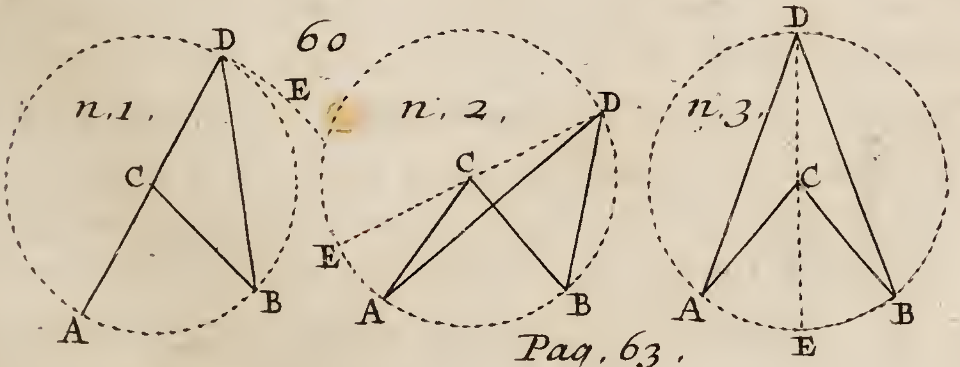
60
n.1.
D
E
C
A
B
n. 2.
D
C
E
A
B
n.3.
D
C
A
B
E

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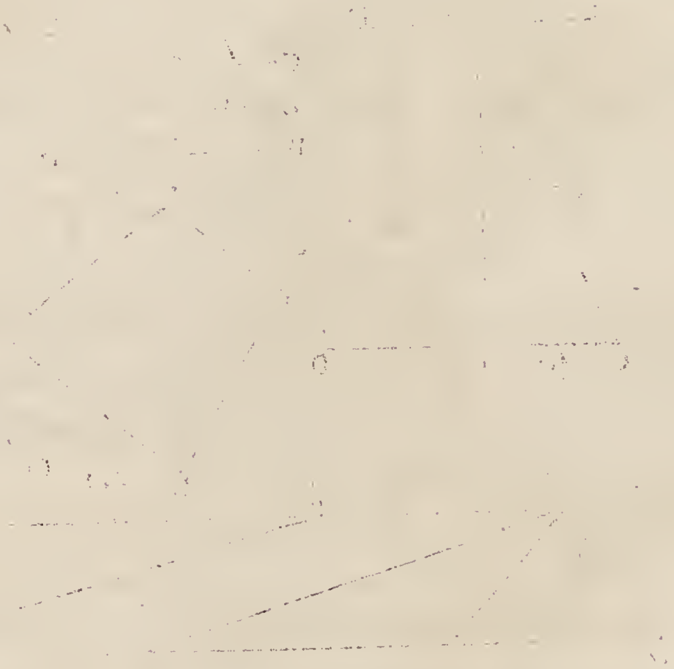

commodate this Logarithmical Artifice to them both. Firſt therefore that Artiſts might aſſign Logarithms to all the common Numbers proceeding from 1 to 1000 and 10000, *&c.* they firſt of all pick out thoſe which proceed in continued Geometrical Proportion, and particularly, tho arbitrariouſly, thoſe which increaſe in a Decuple Proportion, *e.g.* 1. 10 100. 1000. 10000, *&c.*

But now to fit them according to the Foundation of *Conſect.* 3. a Series of Ordinals in Arithmetical Progreſſion, we do'nt only ſubſtitute the ſimple Number 1, 2, 3, *&c.* but augmented with ſeveral Cyphers after them, that ſo we may alſo aſſign their Logarithms in whole Numbers to the intermediate Numbers between 1 and 10, 10 and 100, *&c.* Wherefore, by this firſt Suppoſition, Logarithms in Arithmetical Proportion, anſwer to thoſe Numbers in Geometrical Proportion, after the way we here ſee,

	1	10	100
Log.	0000000	10000000	20000000
		1000	10000
		30000000	40000000, *&c.*

As that they alſo exhibit certain Characteriſtical initial Notes, whereby you may ſee, that all the Logarithms between 1 and 10 begin from 0, the reſt between 10 and 100 from 1, the next from 100 to 1000 from 2, *&c.*

The Logarithms of the Primary Proportional Numbers being thus found, there remain'd the Logarithms of the intermediate Numbers between theſe to be found: For the making of which, after different ways, ſeveral Rules might be given drawn from the Nature of Logarithms, and already ſhewn in *Conſect.* 3. See *Briggs*'s *Arithmetica Logarithmica*, and *Gellibrand*'s *Trigonometria Britannica*; the firſt whereof, *chap.* 5. and the following, ſhews at length both ways delivered by *Neper* in his *Appendix.* But the buſineſs is done more ſimply by *A. Vlacq.* in his Tables of Sines *&c.* whoſe mind we will yet further explain thus: If you are to find, *e. g.* the Logarithm of the Number 9, between 1 and 10, augmented by as many Cyphers as you added to the Logarithm of 10, or the reſt of the Proportionals (*h. e.* between

10000000

10000000 and 100000000) you muſt find a Geometrical Mean Proportional, *viz.* by multiplying theſe Numbers together, and extracting the Square Root out of the Product, by *Prop.* 17. Now if this Mean Proportional be leſs than 9 augmented by as many Cyphers, between it and the former Denary Number you muſt find a ſecond mean Proportional, then between this and that ſame a third; and ſo a fourth, *&c.* but if it be greater, then you muſt find a mean Proportional between it and the next leſs, *&c.* till at length after ſeveral Operations, you obtain the number 9999998, approaching near to 90000000. Now if between the Logarithm of Unity and Ten (*i. e.* between 0 and 10000000) you take an Arithmetical Mean Proportional (05000000) by Biſecting their Sum by Prop. 14. and then between this and the ſame Logarithm of Ten, you take another mean, and ſo a third and a fourth, *&c.* at length you will obtain that which anſwers to the laſt abovementioned, *viz.* 9. See the following Specimen.

A TABLE of the Geometrical Proportionals between 1 *and* 10, *augmented by* 7 *Cyphers, and of the Arithmetical Proportionals between* 0 *and* 10000000, *being the Logarithms correſponding to them.*

Geometrical Mean Proportionals.		*Arithmetical Logar. mean Proportionals*
31622777	Firſt,	05000000
56234132	Second,	07500000
74989426	Third,	08750000
86596435	Fourth,	09375000
93057205	Fifth,	09687500
89768698	Sixth,	09531250
91398327	Seventh,	39609375
90579847	Eighth,	09570312
90173360	Ninth,	09550781
89970801	Tenth.	09541015

Which is thus made: In the firſt Table a Geometrical Mean Proportional between 10000 000 and 100 000 000 s the firſt Number of it; then another Mean between that and the ſame laſt 10000000, gives the ſecond; and ſo to the fifth, 93057205. Which, ſince it is already greater than the Novenary, another Mean between it and the precedent fourth, becomes in order a ſixth, but ſenſibly leſs than the Novenary. Therefore between it and the fifth you will have a ſeventh Mean yet greater than the Novenary; and between the ſixth and ſeventh, an eighth, ſomwhat nearer to the Novenary, but not yet ſenſibly equal, but ſomewhat bigger; moreover between the ſixth and eighth you will have a ninth, between the ninth and ſixth a tenth gradually approaching nearer the Novenary, but yet ſomewhat ſenſibly differing from it. Now if you continue this inquiry of a mean Proportional between this tenth, as ſomewhat too little, and the precedent ninth as ſomewhat too big, and ſo onwards, you will at length obtain the Number 8999 9998, only differing two in the laſt place from the Novenary Number augmented by ſeven Cyphers, and conſequently inſenſibly from the Novenary it ſelf. But for the Logarithm of this in the ſecond Column, by the ſame proceſs you are to find Arithmetical Mean Proportionals between every 2 Logarithms anſwering to every two of the ſuperiour ones, till you find, *e. g.* the Logarithm of the tenth Number 09541015, and ſo at length the Logarithm of the laſt, not ſenſibly differing from the Novenary, 09542425.

Thus having found, with a great deal of labour, but alſo with a great deal of advantage to thoſe that make uſe of them, the Logarithms of ſome of the numbers between 1 and 10, and 10 and 100, *&c.* you may find innumerable ones of the other intermediate Numbers with much leſs labour, *viz.* by the help of ſome Rules, which may be thus obtain'd from *Conſect.* 3 of the precedent Propoſition. *The Sum of the Logarithms of the number Multiplying and the Multiplicand, gives the Logarithm of the Product.* 2. *The Logarithm of the Diviſor ſubtracted from the Logarithm of the Dividend, leaves the Logarithm of the Quotient: the Logarithm of any number doubled, is the Logarithm of the Square, tripled of the Cube, &c.* 4. *The half Logarithm of any number is the Logarithm of the Square Root of that number, the third part of it of the Cube Root, &c.* Thus, *e. g.* if you have found the Logarithm of the number 9, after the way we have ſhewn, by the ſame reaſon you may find the Logarithm of the number 5 (*viz.* by finding mean Proportionals between the ſecond and the firſt number of our Table, and between their Logarithms, *&c.*) and by means of theſe 2 Logarithms you may obtain ſeveral others: Firſt, ſince 10 divided by 5 gives 2; if the Logarithm of 5 be ſubtracted from the Logarithm of 10, you'l have the Logarithm of 2, by Rule the ſecond. Secondly, ſince 10 multiplied by 2 makes 20, and by 9 makes 90, by adding the Logarithms of 10 and 2, and 10 and 9, you'l have the Logarithms of the numbers 90 and 20, by Rule 1. Thirdly, Since 9 is a Square, and its Root 3, half the Logarithm of 9 gives the Logarithm of 3, by Rule 4. ſince 90 divided by 3 gives 30, the Logarithm of this number may be had by ſubtracting the Logarithm of 3 from the Logarithm of 90, by Rule the ſecond. Fifthly, 5 and 9 ſquared make 25 and 81, the Logarithms of 5 and 9 doubled, give the Logarithms of theſe numbers, by Rule 3. In like manner, ſixthly, the Sum of the Logarithms of 2 and 3, or the Difference of the Logarithms of 5 and 30, give the Logarithm of 6, and the Sum of the Logarithms of 3 and 6, or 2 and 9, gives the Logarithm of 18; the Logarithm of 6 doubled, gives the Logarithm of 36, &c. And after this way you may find and reduce into Tables, the Logarithms of Vulgar Numbers from 1 to 10000 (as in the Tables of *Strauch.* p. 182, and the following) or to 100000 (as in the *Chiliads* of *Briggs*) But as to the manner of deducing the Tables of Sines and Tangents from theſe Logarithms of Vulgar Numbers, we will ſhew it in *Schol.* of *Prop.* 55, only hinting this one thing before-hand; that this Artifice of making Logarithms is elegantly ſet forth by *Pardies* in his *Elements of Geometry*, pt 112. by a certain Curve Line thence called the *Logarithmical Line*; by the help whereof he ſuppoſes Logarithms may be eaſily made; and having found thoſe of the numbers between 1000 and 10000, he ſhews, that all others may be eaſily had between 1 and 1000. Wherefore we ſhall Diſcourſe more largely in *Schol. Definit.* 15. *lib.* 2.

Propoſition XXI.

If the firſt Term of never ſo many Continual Proportionals, be ſubtracted from the laſt, and the Remainder divided by the name of the Reaſon or Proportion leſſen'd by Unity, the Quotient will be equal to the Sum of all except the laſt.

Demonſtration.

$e\,a$
$e^2 a$
$e^3 a$
$e^4 a$
$e^5 a$

The laſt Term leſs the firſt $e^6 a - a$

Divided by the name of the Reaſon leſſen'd by unity.

$e - 1$
$e - 1$
$e - 1$
$e - 1$
$e - 1$
$e - 1$
$e - 1$

* *Quote.* $e^5 a + e^4 a + e^3 a + e^2 a + ea + a$; And it is evident from the Operation, that the ſame will always happen tho the number of Terms be continued never ſo far.

CONSECTARYS.

I. Wherefore in adding never ſo great a Series of Geometrical Proportionals, ſince it is enough that the firſt and laſt Term, and the Name of the Reaſon be known, *by this Prop.* and having found at leaſt ſome of the Terms of the Proportion, any other may be afterwards found, whoſe place will be compounded of the places of the two Antecedent ones, according to *Conſect.* 2. *Prop.* 20. *viz.* by Multiplying the Terms anſwering to the two above-mentioned places, and dividing the Product by the firſt Term; thence it will be very eaſie to add a great Series of Proportionals into one Sum, tho the particular ſeparate Terms remain almoſt all of them unknown.

SCHOLIUM.

THeſe are the ſame Practical Arithmetical Rules concerning Geometrical Progreſſions; for the illuſtration of which *Swenterus in Delic.* has given us ſo many pleaſant Examples, *lib.* 1. *Prop.* 59. and *fol.* Firſt of all, that famous Example is of this kind which relates to the Chequer-work'd Table or Board to fling Dice on, with its 64 little Squares, which Dr. *Wallis* has tranſlated out of the Arabick of *Ebn Chalecan*, into Latin, in *Oper. Mathem. part.* 1. *Chap.* 31. for the illuſtration of which, we have heretofore compoſed an Exercitation, and ſhall here only note theſe few things: If there are ſuppoſed 64 Terms of double Proportion from Unity, and the firſt of them, noted with their local Numbers, are theſe that follow;

1	2	4	8	16	32	64	128
	I	II	III	IV	V	VI	VII

You may have the Term of the 13th place, 8192, by multiplying together the VIth and VIIth place; and the Term of the XXVIth place, by ſquaring or multiplying this new Product again by it ſelf, and moreover the Term of the LIId place, by multiplying that Product again by itſelf; and furthermore the Term of the LIXth place, by multiplication of the number laſt found by the number of the VIIth place, and laſtly, the Term of the LXIIId place (*i. e.* the laſt in the propoſed Series) by multiplying this laſt of all by the number of the IVth place.

II. Moreover you may, by this Art, collect infinite Series of Proportional Terms into one Sum, altho it is impoſſible to run over all the Terms ſeparately, becauſe infinite. *e. g.* in a continued Series of Fractions, decreaſing in a double Proportion $\frac{1}{2}$ $\frac{1}{4}$, $\frac{1}{8}$ $\frac{1}{16}$, $\frac{1}{32}$, *&c. ad infinitum*, if you take them backwards, you may juſtly reckon a Cypher or 0, for the firſt Term (for between $\frac{1}{2}$ and 0 there may be an infinite Number of ſuch Terms) and the infinite Sum of theſe Terms will be preciſely equal to Unity; for ſubtracting the firſt 0, from the laſt $\frac{1}{2}$, and the remainder $\frac{1}{2}$ being divided by the name of the Reaſon leſſened by 1, *i. e.* by I. which divides nothing, the

Quotient $\frac{1}{2}$ is the Sum of all the Terms excepting the laſt, by Prop. 21. and ſo the laſt $\frac{1}{2}$ being added, the Sum of all in that Series will be I. Now if the laſt is not $\frac{1}{2}$ but I, the Sum of all will neceſſarily be 2; if 2 be the laſt, the Sum of all will be 4; in a word, it will be always double the laſt Term.

III. And ſince in this caſe the Sum of all the precedent Terms is equal to the laſt Term, the one being ſubtracted from the other, there will remain nothing, *i. e.* $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-\frac{1}{32}$, &c. in *Infinitum*, is $=0$, and alſo $1-\frac{1}{2}-\frac{1}{4}$, &c. or $2-1-\frac{1}{2}-\frac{1}{4}$. &c. $=0$.

IV. In like manner the Sum of infinite Fractions decreaſing in triple Reaſon in an infinite Series ($\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c.) will be equal to $\frac{1}{2}$: for if from the laſt $\frac{1}{3}$ (again in an inverted Order) you ſubtract the firſt 0, and the Remainder $\frac{1}{3}$ be divided by the name of the Reaſon leſſen'd by Unit, *i. e.* by 2, the Quotient $\frac{1}{6}$ will be the Sum of all the antecedent Terms, and adding to this laſt $\frac{1}{3}$ or $\frac{2}{6}$ the Sum of all will be $\frac{3}{6}$ or $\frac{1}{2}$.

V. Thus an infinite Series of Fractions decreaſing from $\frac{1}{4}$ in a Quadruple Proportion ($\frac{1}{4}+\frac{1}{16}+\frac{1}{48}$ &c.) is equal to $\frac{1}{3}$; for ſubtracting the firſt 0 from the laſt $\frac{1}{4}$, and the remainder $\frac{1}{4}$ being divided by the name of the Proportion, i. e. by 3, you will have $\frac{1}{12}$ the ſum of all except the laſt, and adding alſo the laſt $\frac{1}{4}$ or $\frac{3}{12}$, you'l have the whole Sum $\frac{4}{12}$ or $\frac{1}{3}$.

VI. Thus alſo an infinite Series decreaſing from $\frac{1}{5}$ in a Quintuple Proportion ($\frac{1}{5}+\frac{1}{25}+\frac{1}{125}$, &c.) is equal to $\frac{1}{4}$: $\frac{1}{6}+\frac{1}{36}+\frac{1}{216}$, &c. is equal to $\frac{1}{5}$, &c. and ſo any Series of this kind is equal to a Fraction, whoſe Denominator is leſs by an Unit than the Denominator of the laſt Fraction in that Series.

VII. Generally alſo, any infinite Series of Fractions decreaſing according to the Proportion of the Denominator of the laſt Term, and having a common Denominator leſs by an unit than the Denominator of the laſt Term (*e. g.* $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$, &c. or $\frac{3}{4}+\frac{3}{16}+\frac{3}{64}$, &c. or $\frac{4}{5}+\frac{4}{25}+\frac{4}{125}$, &c.) is equal to Unity, after the ſame way as the Series Conſect. 2. which may be comprehended under this kind, and which may be demonſtrated in all its particular caſes by the ſame method we have hitherto made uſe of, or alſo barely ſubſumed from Conſect. 4, 5, and 6. For ſince $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, &c. is equal to $\frac{1}{2}$; $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$ will be equal to $\frac{2}{2}$, or 1, and ſo in the reſt.

VIII. Particularly the ſum of $\frac{3}{27}+\frac{3}{288}+\frac{3}{1152}$, &c, decreaſing in a Quadruple Proportion, is equal to $\frac{1}{18}$; and the ſum of $\frac{3}{64}+\frac{3}{256}+\frac{3}{1024}$, &c. is equal to $\frac{1}{16}$; and the ſum of $\frac{7}{64}+\frac{7}{512}+\frac{7}{4096}$, &c decreaſing in Octuple Proportion, is equal to $\frac{1}{8}$: For ſubtracting the firſt Term o, and dividing the remainder by the name of the Reaſon leſſen'd by 1, *i. e.* by 3, the Quotient $\frac{1}{72}$ gives the ſum of all except the laſt. This therefore (*viz.* $\frac{3}{72}$) being added, the ſum of all will be $\frac{4}{72}$ or $\frac{1}{18}$: In like manner $\frac{7}{64}$ being divided by the name of the Reaſon leſſen'd by Unity, the Quotient will give $\frac{1}{64}$, and adding the laſt, the ſum of all will be $\frac{8}{64}$ *i. e.* $\frac{1}{8}$. So that hence it is evident, that $\frac{1}{18}-\frac{3}{72}-\frac{3}{288}-\frac{3}{1152}$, &c. or $-\frac{1}{18}+\frac{3}{72}+\frac{3}{288}+\frac{5}{1152}$, &c. *in Infinitum*, will be equal to nothing; alſo $\frac{1}{8}-\frac{7}{64}-\frac{7}{512}-\frac{7}{4098}$ &c. $=0$.

IX. *The Sum of a ſimple Arithmetical Progreſſion* (i. e. *aſcending by the Cardinal Numbers*) *continued from* 1, ad Infinitum, *is Subduple of the Sum of the ſame number of Terms, each of which is equal to the greateſt; or on the contrary, this latter Sum is double of the former.* We might have ſubſumed this in *Conſect.* 4. *Prop.* 16. for, prefixing a Cypher before Unity, it will be a caſe of that *Conſectary*, the Sum of the Progreſſion remaining ſtill the ſame. But that this is true, in an infinite Series beginning from Unity (for in a finite or determinate one, the proportion of the Sum is always leſs than double, tho it always approaches to it, and comes ſo much the nearer by how much greater the Series is) we ſhall now thus Demonſtrate: To the Sum of three Terms, 1, 2, 3, *i. e.* 6, the ſum of as many equal in number to the greateſt, *i. e.* 9, has the ſame Proportion as 3 to 2; but to the ſum of ſix Terms, 1, 2, 3, 4, 5, 6, *i. e.* 21, the ſum of as many equal to the greateſt, *i. e.* 36, has the ſame proportion as 3 to $1+\frac{3}{4}$, that is, as 3 to $2-\frac{1}{4}$, the decreaſe being $\frac{1}{4}$: but to the ſum of 12 Terms, which may be found by *Conſect.* 1. *Prop.* 16. $=78$, the ſum of ſo many equal to the greateſt, *viz.* 144. has the ſame proportion (dividing both ſides by 48) as 3 to $1\frac{30}{48}$, *i. e.* 3 to $1+\frac{1}{2}+\frac{1}{8}$ (for 24 make $\frac{1}{2}$, and the remainder $\frac{6}{48}$ is the ſame as $\frac{1}{8}$) that is, as 3 to $2-\frac{1}{4}-\frac{1}{8}$, the decrement being now $\frac{1}{8}$. Since therefore, by doubling the number of Terms onward, you'l find the decrement to be $\frac{1}{16}$, and ſo onwards in double Proportion; the ſum of an infinite Number of ſuch Terms, in Arithmetical Progreſſion, equal to the

greateſt

be to the ſum of the Progreſſion from 1, *ad Infinitum*, as o $2 - \frac{1}{4} - {}_8 - \frac{1}{16}$, &c. that is, by *Conſect.* 2 and 3, as 3 to $-\frac{1}{2}$, that is, as 3 to $1\frac{1}{2}$, or as 2 to 1. Q. E. D.

X. The Sum of any Duplicate Arithmetical Progreſſion (*i.e.* rogreſſion of Squares of whole numbers aſcending) continued m 1 *ad Infinitum*, is ſubtriple of the Sum of as many Terms al to the greateſt as is the number of Terms: For any ſuch e Progreſſion is greater than the ſubtriple Proportion, but roaches nearer and nearer to it continually, by how much farther the Series of the Progreſſion is carried on. Thus Sum of 3 Terms 1, 4, 9 = 14 is to thrice 9 = 27 as $1\frac{5}{9}$, or 3, or $1 + \frac{1}{2} + \frac{1}{18}$ to 3 (dividing both ſides by 9,) the Sum ix Terms, 1, 4, 9, 16, 25, 36, *viz.* 91. to ſix times , *i. e.* to 216 (dividing both ſides by 72) is as $1 + \frac{1}{4} + \frac{1}{72}$ 3; and the Sum of 12 Terms 650, to 12 times 144, *i. e.* 1728 (dividing both ſides by 576) is as $1 + \frac{1}{8} + \frac{1}{288}$ to *&c.* the Fractions adhering to them thus conſtantly decrea- g, ſome by their half parts, others by three quarters (for $\frac{1}{18}$ $\frac{4}{2}$; therefore the firſt decrement is $\frac{3}{72}$ and $\frac{1}{72}$, is $\frac{4}{288}$; there- the ſecond decrement is $\frac{3}{288}$, &c.) Wherefore the Sum of Infinite Progreſſion will be to the Sum of the like number Terms equal to the greateſt, as

$$\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{8} \\ \quad - \frac{1}{4} - \frac{3}{72} \\ \quad - \frac{1}{8} - \frac{3}{288} \\ \quad - \frac{1}{18}, \&c. \frac{3}{1152}, \&c. \end{array}$$

3, that is, by Conſect. 3 and 8, as 1 to 3. Q.E.D.

XI. The Sum of a triplicate Arithmetical Progreſſion (*i. e.* ending by the Cubes of the Cardinal Numbers) proceeding m 1 thro' 27, 64, &c. *ad Infinitum*, is Subquadruple of Sum of the like number of Terms equal to the greateſt. r the Sum of 4 Terms, 1, 8, 27, 64, *i.* 100, to 4 times , *i.e.* 256 (dividing both ſides by 64) will be found to be $1 + \frac{1}{2} + \frac{1}{16}$ to 4; but the Sum of 8 Terms, 1, 8, , 64, 125, 216, 343, 582, *i. e.* 1296 to 8 times 512, t is, 4096 (dividing both Sides by 1024.) will be found be as $1 + \frac{1}{4} + \frac{1}{74}$ to 4, *&c.* The adhering Fractions thus conſtantly decreaſing, the one by their $\frac{1}{2}$ part, the others by (for $\frac{1}{16}$ is $\frac{4}{64}$, and $\frac{1}{64}$ is $\frac{4}{256}$, &c. Wherefore the Sum of the I finite Progreſſion will be to the Sum of a like (Infinite) nu ber of Terms, equal to the greateſt, as

$$\begin{array}{llll} 1 + \frac{1}{2} & & + \frac{1}{8} & \\ - \frac{1}{4} & & - \frac{3}{64} & \\ - \frac{1}{8}, & \text{\&c.} & - \frac{3}{256}, & \text{\&c. to } 4; \end{array}$$

that is, by Conſect. 2 and 8, as 1 to 4. Q. E. D.

XII. The Sum of an Infinite Progreſſion, whoſe great Term is a Square Number, the others decreaſing according the odd numbers 1, 3, 5, 7, *&c.* is in Subſeſquialteran P portion of the Sum of the like number of equal Terms, *i.* as 2 to 3. For the Sum of three ſuch Terms, *e. g.* 9, 8, *i. e.* 22 to thrice 9, *i. e.* 27. is (dividing both ſides by 9) $2\frac{4}{9}$, *viz.* $\frac{8}{18}$ to 3, or $2+\frac{1}{2}-\frac{1}{18}$ to 3. But the Sum of ſuch Terms, 36, 35, 32, 27, 20, 11, *i. e.* 161, to ſix tim 36, *i. e.* 216 (dividing both ſides by 72) is as $2+\frac{1}{4}-\frac{1}{72}$, 8 the adhering Fractions thus always decreaſing, ſome by $\frac{1}{2}$, thers by $\frac{3}{4}$, as above in Conſect. 10. Wherefore the Sum the Infinite Progreſſion will be to the Sum of the like numb of Terms equal to the greateſt, as

$$\begin{array}{l} 2 + \frac{1}{2} - \frac{1}{18} \\ \quad - \frac{1}{4} + \frac{3}{72} \\ \quad - \frac{1}{8} + \frac{3}{288}, \text{ \&c. to } 3, \end{array}$$

i. e. by Co ſect. 3 and 8, as 2 to 3. Q. E. D.

SCHOLIUM II.

THus we have, after our method, demonſtrated the ch Foundations of the *Science* or *Method*, or *Arithmetick Infinites*, firſt found out by Dr. *John Wallis*, Savilian Profeſ of Geometry at *Oxford*, and afterwards carried further by *De lerus Cluverus*, and *Iſmael Bullialdus*. And from theſe Foun tions we will in the following Treatiſe demonſtrate, and t directly and *à priori*, in a few Lines, the chief Propoſitions Geomet ometry, which the Antients have ſpent ſo much labour, and npoſed ſuch large Volumes to demonſtrate, and that but inectly neither.

Propoſition XXII.

THe Powers of Proportionals whether continuedly or diſcretely, ſuch as the Squares, Cubes, &c. *are alſo Proportional.*

Demonſtration.

	Continual Proportionals.				Diſcrete Proportionals.			
	a	ea	e^2a	e^3a	a	ea	b	eb
ares	aa	e^2a^2	e^4a^2	e^6a^2	a^2	e^2a^2	b^2	e^2b^2
bes	a^3	e^3a^3	e^6a^3	e^9a^3	a^3	e^3a^3	b^3	e^3b^3

Q. E. D.

SCHOLIUM.

YOu founded in this Truth, 1. the Reaſon of the Multiplication and Diviſion of Surd Quantities: For ſince from Nature and Definition of Multiplication, it is certain, that s to the Multiplier as the Multiplicand to the Product (for multiplicand being added as many times to it ſelf as there Units in the Multiplier, makes the Product) if the $\sqrt{5}$ is be multiplied by $\sqrt{3}$, then as 1 to the $\sqrt{3}$, ſo the $\sqrt{5}$ the Product; and, by the preſent Propoſition, as 1 to 3, ſo to the □ Product, *i. e.* to 15. Whereſore the Product is 15; and ſo the Rule for Multiplying Surd Quantities is this: *ltiply the Quantity under the Radical Signs, and prefix a Radical n to the Product.* Likewiſe ſince it is certain m the Nature of Diviſion, that the Diviſor is the Dividend as 1 to the Quotient (for the uotient expreſſes by its Units how many times Diviſor is contained in the Dividend) if the $\sqrt{15}$ is to be vided by $\sqrt{5}$, you'l have $\sqrt{5}$ to the $\sqrt{15}$ as 1 to the Quotient, and, by the preſent Scholium, 5 to 15, as 1 to the □ of Quotient, *i. e*, to 3. Therefore the Quotient is the Root 3, and ſo the Rule of dividing Surd Quantities this; *viz.*

(*a*) *Eucl. lib.* 6. *prop.* 22.

Divide the Quantities themſelves under the Radical Signs, and
fix the Radical Sign to the Quotient.

II. Hence alſo flows the uſual Reduction in the Arit
tick of Surds, of Surd Quantities to others partly Rational
on the contrary, of thoſe to the form of Surds, *e. g.* If
would reduce this mixt Quantity $2a\sqrt{b}$, *i. e.* $2a$ multiplie
the $\sqrt{b}$, to the form of a Surd Quantity; which ſhall all be
tained under a Radical Sign; The Square of a Rational C
tity without a Sign $4aa$, if it be put under a Radical Sig
this form $\sqrt{4aa}$, it equivalent to the Rational Quantity
but the $\sqrt{4aa}$ being multiplied by $\sqrt{b}$ makes $\sqrt{4aab}$. by
1. of this Scholium. Therefore $\sqrt{4aab}$ is alſo equivalent
Quantity firſt propoſed $2a\sqrt{b}$. Reciprocally therefore, i
form of a meer Surd Quantity $\sqrt{4aab}$, is to be reduced t
more Simple, which may contain without the Radical
whatever is therein Rational, by dividing the Quantity
prehended under the ſign √ by ſome Square or Cube, &
here by $4aa$, (*i. e.* $\sqrt{4aab}$ by $\sqrt{4aa}$, i. e. $2a$) the Quotient
be $\sqrt{b}$, which multiplied by the Diviſor $2a$, will rightl
preſs the propoſed Quantity under this more ſimple Form 2
Which may alſo ſerve further to illuſtrate the Scholia of
7. and 10.

Propoſition XXIII.

IF there are four Quantities Proportional, (a, ea, b, eb)
will be alſo Proportional,

(α) *Eucl.* 15, 16. v. 9, 10, 13, vi. (β) 18, v. (γ) 17, v.

1. Inverſly. ea to a as eb to b.
2. Alternatively, (α) a to b as ea to eb.
3. Compoundedly, (β) $a+ea$ to ea, ſo b
to eb.
4. Converſly, $a+ea$ to a as $b+eb$ to b.
5. Dividedly, (γ) $a-ea$ to $\left(\begin{array}{l} ea \text{ as } b-eb \text{ to} \\ \text{or } a \end{array}\right.$
6. (α) By a Syllepſis, a to ea as $a+b$ to $ea+eb$.
7. By a Dialepſis, a to ea as $a-b$ to $ea-eb$.

Which are all manifeſt, by comparing the Rectangles
Means and Extremes according to to *Prop.* 19. and its *Con*

or by dividing any of the Conſequents by their Antecedents, according to *Def.* 31.

Propoſition XXIV.

IF in a (β) double Rank of Quantities you have
as a to ea, and alſo as ea to oa,
ſo b to eb, ſo eb to ob,
then you'l have alſo by proportion of Equality orderly placed,
as the firſt a, to the laſt oa, in the firſt Series;
ſo the firſt b, to the laſt ob, in the ſecond Series.

Which is manifeſt from the Terms themſelves.

Propoſition XXV.

BUt (γ) if they are diſorderly plac'd
as oa to ea) * as ea to a †
† ſo eob to ob) ſo ob to eb, * you'l have here
again by proportion of Equality,
as the firſt oa to the laſt a, in the firſt Series;
ſo the firſt eob to the laſt eb, in the ſecond Series.

As is evident from the Rectangles of the Extremes and Means, as alſo from the very Terms.

(α) *Eucl.* 1. 12. v. 5. 6. 12. vii.

(β) *Eucl.* 3, 20, 22. *lib.* v. 14. vii.

(γ) *Eucl.* 21, 23. *lib.* v.

Propoſition XXVI.

IF (α) as the whole ea to the whole a, ſo the part eb to the part b; then alſo will
the Remainder Remainder Whole Whole
$ea-eb$ to the $a-b$, as the ea to the a.

This is evident from the Rectangle of the Extremes and Means, both which are $eaa-eab$. Q.E.D.

Propoſition XXVII.

REctangles or Products having one common Efficient or Side, are one to another as the other Efficients or Sides.

Demonſtration.

Suppoſe the Products to be $a\,b$ and $a\,c$, having the common Efficient a; I ſay they are

as b to c, ſo $a\,b$ to $a\,c$.

Which is evident at firſt ſight, by comparing the Products of the Extremes and Means, and alſo fully ſhews, that other way of proving Proportionality, whereby by dividing the Conſequents by their Antecedents, the identity or ſameneſs of the Quotients are wont to be demonſtrated.

SCHOLIUM I.

I. THe Reduction of Fractions either to more compounded or more ſimple ones is founded on this Theorem; on the one hand by multiplying, on the other by dividing, by the ſame quantity, both the Numerator and the Denominator, as, *e. g.* $\frac{c}{b}$ and $\frac{a\,c}{ab}$ and $\frac{eac}{eab}$, $\frac{1}{3}$, $\frac{3}{9}$, $\frac{4}{12}$, *&c.* are in reality the ſame Fractions. And

II. The Reduction of Fractions to the ſame Denomination, as if $\frac{b}{c}$ and $\frac{a}{d}$ are to be changed into two others that ſhall have ſame Denominator; this is to be done by multiplying the Denominators together for a new Denominator, and each Numerator by the Denominator of the other for a new Numerator, and you'l have for the two Fractions above ———— $\frac{b\,d}{c\,d}$ and $\frac{a\,c}{c\,d}$.

(α) *Eucl.* 5 & 19. *lib.* v. 7 & 11. *l.* 7. (β) *Beſides ſeveral other Prop. ſee alſo the* 17 & 18 *lib.* vii.

SCHOLIUM II.

WE will here for a concluſion of Proportionals, ſhew the way of cutting or dividing any Quantity in *Mean and Extreme Reaſon, viz.* if for the greater Part you put x, the leſs will be $a-x$; and ſo by Hypoth. theſe three, a, x and $a-x$, will be proportional, by *Def.* 34. Therefore by *Prop.* 17. the Product of the Extremes $aa-ax=$ to the Square of the Mean xx, and (adding on both ſides ax) $aa=xx+ax$; and moreover adding on both ſides $\frac{1}{4}aa$, you'l have $\frac{5}{4}aa=xx+ax+\frac{1}{4}aa$. Now this laſt Quantity, ſince it is an exact *S*quare, whoſe Root is $x+\frac{1}{2}a$, you'l have $\sqrt{\frac{5}{4}aa}=x+\frac{1}{2}a$, and (ſubtracting from both ſides $\frac{1}{1}a$) $\sqrt{\frac{5}{4}aa}-\frac{1}{2}a=x$.

Now therefore we have a Rule to determine the greater part of a given Quantity to be divided in Mean and Extreme Reaſon, *viz.* if the given Quantity be a Line, *e. g.* $AB=a$ (*Fig.* 58.) join to it (α) at Right Angles $AC=\frac{1}{2}a$: Wherefore by the Theorem of *Pythagoras* from *Schol. Definit.* 13. the Hypothenuſe CB, or, which is equal to it, $CD=\sqrt{\frac{3}{4}aa}$; and conſequently $AC=\frac{1}{2}a$ being taken out of CD, the Remainder AD, or AE, which is equal to it, will be $=x$, the greateſt part ſought; according to *Euclid*, whoſe Invention this firſt Specimen of Analyſis, by way of Anticipation, reduces to its original Fountain. As for Numbers (tho none accurately admits of this Section) the ſenſe of the Rule, or which is all one as to the thing it ſelf, is this: Add the Squares of a whole Number and its half, and ſubtract the ſaid half from the Root of the Sum (which can't be had exactly, ſince it is $\sqrt{\frac{5}{4}}$.

(α) *Eucl.* 11. *lib.* 11. & 30. *lib.* vi.

CHAP. V.

CAAP. V.

Of the Proportion or Reaſons of Magnitudes of the ſame kind in particular.

Propoſition XXVIII.

Triangles and Parallelograms, alſo Pyramids and Priſms and Parallelepipeds, laſtly Cones and Cylinders, each kind compared among themſelves, if they have the ſame Altitude, are in the ſame Proportion to one another as their Baſes.

Demonſtration.

This and the following Propoſition might have been by a bare Subſumption added, as *Conſectarys* to the precedent; for the Altitudes in the one, and Baſes in the other, may be looked on as common Efficients, and the Magnitudes mentioned as their Products: But for the greater diſtinction ſake, we will thus Demonſtrate them more particularly.

I. If the equal Altitudes of two Triangles, or (α) two Parallelograms, are called *b* and the Baſe of the one *a*; and of the other *ea*; theſe Products will be *ba* and *bea*, the other $\frac{1}{2}$ *ba* and $\frac{1}{2}$ *bea*, by *Def.* 28. *Schol.* 2.

II. Likewiſe the equal Altitudes of two Priſms (β) or Pyramids, may be called *b*, and the Proportion of their Baſes expreſſed by *a* and *ea*; and the Priſms will be among themſelves as *ba* to *bea*, and the Pyramids as $\frac{1}{3}$*ba* to $\frac{1}{3}$*bea*, by the ſaid *Schol. Num.* 3.

III. There is alſo the ſame Proportion of Cylinders and Cones as of Pyramids and Priſms, by *Conſect.* 4. *Definit.* 17. But,

(α) *Eucl. Prop.* 1. *lib.* vi.
(β) *Prop.* 5. 6. *lib.* xii. 25, 32. xi. & *Conſ.* 30 & 31 *of the ſame*
(γ) *Prop.* 11. *lib.* xii.
(δ) *Prop.* 35, 36, 37, 38, 39, 40, *lib.* *lib.* 1 & 29, 30, 31 *lib.* xi.

as a to ea ſo is ba to bea.
—$\frac{1}{2}ba$ to $\frac{1}{2}bea$.
—$\frac{1}{3}ba$ to $\frac{1}{3}bea$. Q. E. D.

CONSECTARY.

THerefore Magnitudes of the ſame kind upon the ſame or equal Baſes (♪) and of the ſame heighth, are equal among themſelves, and the contrary.

Propoſition XXIX.

TRiangles and Parallelograms, Pyramids and Priſms and Parallelepipeds, Cones and Cylinders, being on equal Baſes, are in the ſame Proportion as their heighths. (*)

Demonſtration.

Let all their Baſes be called a, and the Proportions of their Heighths be as b to eb: Therefore, 1. the Parallelograms, Parallelepipeds and Cylinders, are one to the other of the ſame kind, as ba to eba; the Triangles as $\frac{1}{2}ba$ to $\frac{1}{2}eba$; the Pyramids and Cones as $\frac{1}{3}ba$ to $_3 eba$, by *Def.* 28. *Schol.* 2. But,
as b to eb, ſo is ba to eba.
and $\frac{1}{2}ba$ to $\frac{1}{2}eba$.
and $\frac{1}{3}ba$ to $\frac{1}{3}eba$. Q. E. D.

Propoſition XXX.

EQual (‖) *Triangles, Parallelograms, Priſms, Parallelepipeds, alſo equal Pyramids, Cones, and Cylinders, have their Baſes and Heighths reciprocally Proportional.*

Demonſtration.

For if for the equal Triangles you put $\frac{1}{2}ab$, for the Cones and Pyramids $\frac{1}{3}ab$, and for the reſt ab;

whether the Baſes of the equal Quantities are ſuppoſed to be a, and ſo the Altitudes

(*) *Schol prop.* 1.*l.*6,12,13,14
‖*lib.*6.*prop.*14. 15.*l.*11.34. *l.* 12.11, *and its Coroll. alſo Prop.* 15.

on both ſides b; or if the Baſe of the one be a and b the Altitude, but the Baſe of the other b and the Altitude a, you'l certainly have eitherways,

as a to a, ſo Reciprocally b to b;

the Baſe of the former to the Baſe of the latter, as the Altitude of the latter to the Altitude of the former, or,

as a to b, ſo Reciprocally a to b. Q.E.D.

CONSECTARY.

AND thoſe Magnitudes of the ſame kind, whoſe Baſes and Altitudes are thus Reciprocal, are equal by *Prop.* 18. for the Product or Rectangle of the Extremes is ab, and that of the Means ba.

Propoſition XXXI.

TRiangles, Parallelograms, Priſms, Parallelepipeds, Pyramids, Cones and Cylinders, each kind compared among themſelves, are in the Proportion compounded of the Proportion of their Altitudes and Baſes. (a)

Demonſtration.

Suppoſe the Baſe of the one to be a, and the other ea, and the Altitude of the one b, of the other ib; therefore the one will be to the other,

as ab to $eiab$,

or $\frac{1}{2}ab$ to $\frac{1}{2}eiab$,

or $\frac{1}{3}ab$ to $\frac{1}{3}eiab$; *i. e.* every where as a to eia, *i. e.* in Proportion compounded of a to ea, and of b to ib, by *Conſect.* 2. *Def.* 34. Q.E.D.

SCHOLIUM.

FRom what we have hitherto Demonſtrated, we may not only make an eſtimate of Magnitudes of the ſame kind compared together, which is eaſie to any one who attentively conſiders them; but alſo with *F. Morgues*, deduce a General Rule of expreſſing the Proportions of any Rectilinear Planes or Solids, contained under

(a) *Prop.* 23. *lib.* 6.

Plane Surfaces, by the proportion of one Right Line to another. For ſince the one may be reſolved into Triangles, and the other into Pyramids, having firſt two Rectilinear Planes given and thus reſolved, upon a Right Line I make the $\triangle$ *abc* (*Fig.* 49.) Equal to one of the Triangles of either of the Planes *e. g.* to ABC; then having drawn the Parallel *cm*, if the $\triangle$BCD has the ſame Altitude with the former, you need only joyn the Baſe *B*C to the Baſe *a b*. But if the Altitude DS is greater than the Altitude of the other *e. g.* by $\frac{1}{5}$, then you muſt make *bf* equal to the Baſe *bc* augmented by a fifth part, and the Triangle *bcf* will $=$BCD, and the whole *acf* $=$ to the Rectilinear Figure ABCD. If now therefore I likewiſe make another Triangle *ghi* equal to another Rectilinear Figure between the ſame Parallels, then will the $\triangle$ *a c f* be to the $\triangle$ *g h i*, that is, the Right Lined Figure ABCD to the Right Lined Figure FGHIK, as *af* to *gh*, by *Prop.* 28. 2. Having 2 Right Lined Solids given, and having reſolved them into Triangular Pyramids, they may be transferr'd between 2 parallel Planes, *viz.* by augmenting or diminſhing their Triangular Baſes reciprocally, according to the exceſs or defect of their Altitudes, as was done above with the Linear Baſes; then thoſe Triangular Baſes on both ſides may be converted into one Triangular Baſe, and conſequently each Solid into a Pyramid equal to it ſelf; which two Pyramids will be one to the other as their Triangular Baſes. And becauſe the Proportions of theſe Baſes may be reduced to the Proportion of two Lines each to the other, by N° 1. of this; therefore alſo the Reaſon or Proportion of the two Solids may be expreſſed by the Proportion of two Lines. Q.E.D.

Propoſition XXXII.

CIrcles (β) *are in the ſame Proportion to one another as the Squares of their Diameters.*

Demonſtration.

Suppoſe *a* to be the Diameter of one Circle, and *b* of another; then by *Definit.* 31. *Conſect.* 1, the Area of the one will be $\frac{1}{4}$ *eaa*, and that of the other $\frac{1}{4}$ *ebb*. But as *aa* to *bb* ſo is $\frac{1}{4}$ *eaa* to $\frac{1}{4}$ *ebb*, by *Conſect.* 1. *Prop.* 19. Q. E. D.

(β) *Eucl. Prop.* 2. *l.* 12.

CON-

CONSECTARY I.

THe ſame will in like manner be manifeſt of like Sectors of Circles, while for the parts of the Periphery you put ia and ib, as for the wholes we put ea and eb: for thus the Area of the one will be $\frac{1}{4}iaa$, and of the other $\frac{1}{4}ibb$.

CONSECTARY II.

CYlinders whoſe Altitudes are equal to the Diameters of their Baſes, are in proportion to one another as the Cubes of their Diameters; for the Cylinders will be $\frac{1}{4}ea^3$ and $\frac{1}{4}eb^3$, the Cubes a^3 and b^3.

CONSECTARY III.

HEnce alſo (whatever the Reaſon of the Sphere is to the (α) Cylinder of the ſame Diameter and Heighth; which we will hereafter Demonſtrate, and which in the mean while we will denote by the name of the Reaſon y) I ſay, hence Spheres which have the ſame Proportion to one another as theſe Cylinders (*viz.* as $\frac{1}{4}ea^3$ to $\frac{1}{4}eb^3$, ſo $\frac{1}{4}yea^3$ to $\frac{1}{4}yeb^3$) will alſo (by *Conſect.* 1.) be in the ſame proportion as the Cubes, a^3 to b^3; as is alſo evident from theſe Terms themſelves.

Propoſition XXXIII.

THE Angle (β) at the Center of any Circle ACB (Fig. 60) is to an Angle at the Circumference which has the ſame Arch for its Baſe ADB, as 2 to 1.

Demonſtration.

The truth of this has already appear'd from *Schol. Definit.* 10 N° 3. but here we will demonſtrate it otherwiſe in its three Caſes, after *Euclids* way. In the firſt Caſe DE being conceived Parallel to CB, by *Def.* 11. *Conſect.* 1 and 2. the External Angle ACB is = to the Internal An-

(α) *Prop.* 18 *lib.* 12.

(β) *Eucl. Prop.* 20. *l.* 3.

gle ADE, and the Angle BDE, is equal to the alternate Angle BCD, *i. e.* to the other at the Baſe CDB, by *Conſect.* 2. *Definit.* 13. Therefore BDE is as 1, and CDE, *i. e.* A C B as 2.

In the ſecond Caſe the whole ECB is double of the whole EDB, and the ſubtracted Angle ECA is double of the ſubtracted Angle EDA, by Caſe 1. Therefore the Remainder ACB is alſo double of the Remainder ADB, by Prop. 26. In the third Caſe the part ECA is double of the part EDA, and alſo the part ECB is double of the part EDB, by Caſe 1. Therefore the whole ACB is double the whole ADB. Q.E.D.

CONSECTARYS.

I. HEnce all Angles ADB (α) in the ſame Segment are equal, and the Angle ADB (*Fig.* 61.) in a Semicircle is a Right one; becauſe the Aperture at the Center anſwering to it, ACB contains two Right Angles: The Angle in a leſs Segment than a Semicircle EDF, is greater than a Right one; becauſe the Aperture at the Center EGHFC anſwering to it, comprehends more than two Right Angles. An Angle, laſtly, in a Segment greater than a Semicircle GDH, is leſs than a Right one; becauſe its double at the Center GCH is leſs than two Right ones. All which we have already otherwiſe demonſtrated in *Schol. Def.* 10. N° 6.

II. All the three Angles (β) of any Triangle ABD taken together, are equal to two Right ones; becauſe they are the half of the three at the Center C, which always make 4 Right ones, by *Definit.* 8. *Conſect.* 2.

(α) *Eucl. prop.* 21. 27. 31. *lib.* 3.
(β) *prop.* 32. *l.* 1.

III. Therefore any external Angle IAB, is equal to the two Internal oppoſite ones at B and D; becauſe that, as well as they with the other contiguous to them BAD, make two Right ones, by *Conſect.* 1. of the ſame *Definit.*

IV. And the greateſt Side of a Triangle, becauſe it inſiſts on a (α) greater Arch of a Circumſcribed Circle, does alſo neceſſarily ſubtend a greater Angle, by vertue of *Conſect.* 1. hereof.

Propoſition XXXIV.

IN *Equiangular Triangles* (*ACB and* abc, *Fig.* 62.) *the Sides about the equal Angles are Proportional,* viz. *as AB to BC, ſo* ab *to* bc, *and as BC to CA ſo is* bc *to* ca. *&c.* (β)

Demonſtration.

For having deſcribed Circles thro' the Vertex of each Triangle, according to *Conſect.* 6. *Definit.* 8. by reaſon of the ſuppoſed equality of the Angles A and a, B and b, C and c, the Arches alſo AB and ab, &c. will neceſſarily agree in the number of Degrees and Minutes, by the foregoing 33 *Prop.* and ſo alſo the Chords AB and ab, BC and bc, &c. will agree in the number of Parts of the Radius or whole Sine ZA and za, by *Conſect.* 2. *Definit.* 10. Wherefore as many ſuch Parts as AC has, whereof az has alſo 10000000, ſo many ſuch alſo will ac have, whereof az has alſo 10000000, *&c.* Therefore AC is to CB as ac to cb, &c. Q. E. D.

CONSECTAYS.

I. WHerefore by the ſame neceſſity the Baſes of ſuch Triangles AB and ab, will be proportional to their Altitudes CD and cd, as being Right Sines of the like Arches CB and cb, or rather CE and ce; and ſo for ſimilar or like Triangles (and conſequently alſo Parallelograms) we may rightly ſuppoſe that their Baſes are as a to ea, and their Heighths as b to eb; tho we muſt not immediately conclude on the contrary, that becauſe their Baſes and Altitudes are ſo, therefore they are Similar.

II. As alſo in Similar Parallelepipeds it will be manifeſt to any attentive Perſon, that the Baſes are in a duplicate Proportion of the Altitudes. For ſince the Planes of Similar Solids are equal in number, and Similar each to the other, if for 𝔄𝔅 (*Fig.* 63.) we put a, and for 𝔅ℭ b, AB will $=$ ea and BC $=$ eb; and ſo that Baſis will be to this as ab to $eeab$. Moreover, having let fall the Perpendiculars EH and 𝔈ℌ, the Triangles

FBH and 𝔈𝔅ℌ are ſimilar, and by putting *c* for 𝔅ℭ, BE will be *e c*, putting alſo *d* for 𝔈ℌ, EH will conſequently be *e d.* But the Reaſon of the Baſe *a b* to the Baſe *e e a b*, is duplicate of the Reaſon of *d* to *e d*, by *Def.* 34. Wherefore in Similar Parallelepipeds we may rightly ſuppoſe, that their Baſes are as *a b* to *eeab*, or as *a* to *eea*, and their Altitudes as *d* to *ed.*

SCHOLIUM I.

FRrom this Propoſition flows firſt of all the chiefeſt part of Trigonometry for the Reſolution of Right Angled ΔΔ: For ſince in any Right Angled Triangle, if one ſide, *e. g.* AB (*Fig.* 64.) be put for the whole Sine, the other BC will be the Tangent of the oppoſite Angle at A (and in like manner if CB be the whole Sine, BA will be the Tangent of the Angle C;) but if the Hypothenuſe AC be made Radius or whole Sine, then the Side BC will be the Right Sine of the Angle A, or the Arch CD deſcribed from the Center A, and AB the Right Sine of the Angle C, or the Arch AE, deſcribed from the Center C, (we will omit mentioning the Secants, becauſe the buſineſs may be done without them) which all follow from *Def.* 10. Wherefore you may find,

I. The Angles.

1. From the Sides	by inferring	As one leg to the other, ſo the whole Sine to the Tangent of the Angle oppoſite to the other Leg.
2. From the Hypoth. & one ſide.		As the Hyp. to the W.S.(whole ſine) ſo the given leg to the S. of the opp. angle

II. The Sides.

1. From the Hypoth. and Angles:	As the W. S. to the Hypoth. ſo the Sine of the Angle, oppoſite to the Leg ſought, to the Leg it ſelf.
2. From one Leg and the Angles:	As the W. S. to the given Leg, ſo the Tangent of the Angle adjacent to it, to the Leg ſought.
3. From the Hypoth. and one of the Sides:	Having firſt found the Angles, it's done by the 2, 1. or by the *Pythagorick* Theorem.

III. The Hypothenuſe.

1. From the Angles and one of the Legs.	As the S. of the Angle, oppoſite to the given Leg, to that Leg, ſo the W. S. to the Hypoth.
2. From the Legs given;	Having firſt found the Angles its done by the 1. or by the *Pythagorick* Theorem.

III. Inverſly alſo, if two Triangles ABC and 𝔄𝔅ℭ (i the Figure of the preſent Propoſition) have one Angle of one equal to one Angle of the other (*e. g.* B and 𝔅) and the *Side* that contain theſe equal Angles proportional (*viz.* as AB to BC ſo 𝔄𝔅 to 𝔅ℭ) then the other Angles (A and 𝔄, C and ℭ will be alſo equal, and the Triangles ſimilar (α) for to the like Chords AB and 𝔄𝔅, BC and 𝔅ℭ, there anſwer by the Hypoth. like or ſimilar Arches, *i. e.* equal in the number o Degrees and Minutes; and to theſe alſo there anſwer equal Angles both at the Periphery and Center.

IV. (*Fig.* 65. N° 1.) If (β) the Sides of the Angle BAC are cut by a Line DE, parallel to the Baſe BC, the Segments o thoſe Sides will be proportional, *viz.* AE to EC as AD to BD for by reaſon of the Paralleliſm of the Lines BE and BC, th Triangles ADE and ABC are Equiangular: Therefore as th whole BA to the whole AC, ſo the part AD to the part AE and conſequently alſo the remainder EC to the remainder DB as the part EA to the part AD, by *Prop.* 26. and alternativel by *Prop.* 24. EC will be to EA as BD to AD.

SCHOLIUM II.

THere are ſeveral uſeful Geometrical Practices depend o this *Conſectary* and its *Propoſition.* 1. That (γ) whereb we are taught to cut off any part required, *e.* ⅓ from a given Line AB, and ſo generally to cu or divide any given Line AC, in the ſame pro portion as any other given Line, is ſuppoſed t be divided in D, (and conſequently into as man equal parts as you pleaſe;) *viz.* if in the fir Caſe, having drawn any Line AF, you take AD 1, and make DB 2, and having joined CB, dra

th

(α) *Eucl prop.* 6. *lib.* 6.

(β) *Eucl.* 2. *lib.* 6.

(γ) *Eucl. l.* 9 & 10. *l.* 6.

the Parallel LE: for as AD to DB ſo is AE to AC; that is, as 1 to 2, by this 4th *Conſect.* therefore AE is one third of the whole AC, *&c.*

II. A Rule (α) to find a third Proportional to the 2 Right Lines given AB and BC (N° 2. *Fig.* 65.) (or a fourth to three given;) if, *viz.* having drawn AF at pleaſure, you make AD equal to BC, and Joining DB, draw the Parallel EC: For as AB to BC, ſo AD *i. e.* BC) to DE. Now if AD be not equal to BC but to another (*viz. a*) third Proportional, then by the ſame Reaſon DE will be a fourth Proportional.

III. Another Rule (β) to find a mean Proportional between two Right Lines given AC and CB; which is done by joining both the Lines together, and from the middle of the whole AB deſcribing a Semicircle, and from C erecting the Perpendicular CD: For ſince the Angle ADB is a Right one, by *Conſect.* 1. of the preceding *Propoſition*, and the two Angles at C are Right ones, and thoſe at A and B common to the whole Triangle ADB, and to the two partial ones ACD and BCD, theſe two will be Equiangular and Similar to the great one, and conſequently to one another: Therefore by the preſent *Propoſition*, as AC to CD, ſo CD to CB, Q. E. D. and alſo as AB to BD ſo BD to BC, and as AB to AD ſo AD to AC, *&c.*

IV. The Analytical Praxis of multiplying and dividing Lines by Lines, ſo that the Product or Quotient may be a Line; and alſo the way of Extracting Roots out of Lines: Which *Des Cartes*, gives us, p. 2. of his Geom. *viz.* aſſuming a certain Line for Unity, *e. g.* AB (in *Fig.* 65. N° 2.) if AC is to be multiplied by AD, having joined BD, and drawn the Parallel CE, the Product will be AE; for it will be as 1 to the Multiplier AD, ſo the Multiplicand AC to the Product AE; or if AE is to be divided by AC, having joined EC and drawn the Parallel BD, the Quotient will be AD; (for AC the Diviſor, will be to AE the Dividend, as an Unit AB to the Quotient AD;) all which are evident from the Nature of Multiplication and Diviſion, and the

(α) *Eucl.* 11 & 12, *l.* 6.
(β) *Eucl.* 13. *lib.* 6 & *and Eucl.* 8. *lib.* 6.

Precedent Praxes. As alſo taking CB (in the ſame *Fig.* N° 3 for Unity, if the Square Root is to be extracted out of any o ther Line AC, this being joined to your Unity in one Line A and having deſcribed thereon a Semicircle, the Perpendicul CD will be the Root ſought, as being a Mean Proportional b tween the two Extremes CB and AC, according to *Prop.* 17.

V. A Right Line AG which divides (α) any given Angle into two equal Parts (*Fig.* 66.) being prolonged, divides th Baſe BC proportionally to the Legs of the Angle AB and AC For having prolonged CA to E, ſo that AE ſhall be = to AB the Angles ABE and AEB will be equal, by *Conſect.* 2. *Def.* 1 and conſequently alſo equal to each of the halves of the exter nal Angle CAB, by *Conſect.* 3. of the antecedent Propoſitio Therefore the lines AG and EB will be parallel, by *Conſ.* 1. *Def.* 1 Therefore as AC to AE, *i.e.* to AB, ſo GC to GB, by *Co ſect.* 3. of this Propoſition. Q.E.D.

VI. Hence alſo there follows further, by converſion of th laſt inference, as AC+AB to AC, ſo GC+GB (*i. e.* BC) t GC; and inverſly GC to BC as AC to AC+AB; and laſtl alternatively, GC to AC as BC to AC+AB.

N. B. This laſt Inference follows alſo immediately from th preceding *Conſectary.* For by reaſon of the Similitude of th △△ ACG and ECB, as GC to AC ſo BC to CE, *i. e.* to A +AB.

SCHOLIUM III.

FRom theſe two laſt Conſectarys there ari theſe or two or three Practical Rules, th firſt whereof ſhews, how *having the two Legs A and AC given, and alſo the Baſe BC, to find the Se ments GC and GB, made by the Biſection of the Intercrural Angl* (*viz.* by this inference, according to *Conſect.* 6. As the Sum the Sides to one Side (*e. g.*) AC:) ſo the Sum of the Segmen of the Baſe, *i. e.* the whole Baſe to one of the Segments, *vi* that next the ſaid Side GC. 2. It ſhews on the contrary, how *having the Baſe and one of its Segments given, and moreover the Su of the Sides, to find ſeparately the Side AC next the known Segment*

(α) *Eucl.* 3. *lib.* 6.

b

y inferring as the Sum of the Segments, or the Baſe BC to the um of the Sides, ſo the given Segment GC to the ſought AC: r alſo, 3dly, *Having only the Baſe and Sum of the Sides given*, but ot the Segment GC, yet to expreſs its Proportion to the next de AC, —— —— *viz.* in the Quantities of the given erms, by putting (by *Conſect.* 6.) for GC the value of the aſe BC, and for AC the value of the Sum AB+AC; the reat uſe of which laſt Rule will appear hereafter in the Cy- lometry (or Quadrature of the Circle) of *Archimedes*.

VII. In any Triangle ABC (*Fig.* of the preſent *Propoſition*) e Sides are to one another as the Sines of their oppoſite An- les: For they are as the Chords of the double Angles at the enter, by *Prop.* 33. therefore they are alſo one to another as alf thoſe Chords, *i. e.* by *Definit.* 10. as the Sines of the half ngles.

SCHOLIUM IV.

Hence flow two new Rules of Plane Trigonometry, for Oblique-angled Triangles to find, *viz.*

I. The other Angles:

rom 2 gi- en Sides, & n Angle op- poſite to one f them :	by inferring	As the Side oppoſite to the given Angle to the other Side, ſo is the Sine of the given Angle to the ſine of the angle oppoſite to the other Side; which being given, the third is eaſily found.

II. The other Sides:

rom one de and the ngles given,	As the Sine of the Angle oppoſite to the given ſide, to that ſide; ſo is the Sine of the Angle oppoſite to the ſide ſought to the ſide ſought.

So that this way we have reduced all the Caſes excepting one f Plane Trigonometry, and conſequently all *Euthymetry* to eir original Foundations (for in that Caſe of having two des, and the included Angle given, we may find the reſt by the Reſolution of the Obliqueangled Triangle into two Right Angled ones; and ſo it's done by the Rules we have deduc'd in (*Schol.* 1.) I ſay, excepting one, in which from the three ſides of an Obliqueangled Triangle given, you are required to find the Angles: the Rule to reſolve which we will hereafter deduce in the 2d *Conſect.* of *Prop.* 45. from that Theorem which *Euclid* gives us, *lib.* 2. *Prop.* 13.

VIII. Becauſe in in the Right Angled △ BAC (*Fig.* 67.) BC is to CA as CA to CD, by N° 3. of the 2d *Schol.* of this *Prop.* the □ of CA will be = ▭ CE, by *Prop.* 17. In like manner becauſe as CB to BA ſo is *B*A to BD; the □ of BA will be = to ▭ BE: Wherefore the two Rectangles BE and CE taken together, that is, the □ of the Hypothenuſe BC, will be = to the two □'s BA and CA taken together: Which is the very Theorem of Pythagoras demonſtrated two other ways in *Schol.* of *Definit.* 13.

SCHOLIUM V.

THis Theorem of *Pythagoras* as it furniſhes us with Rules of adding Squares into one Sum, or ſubtracting one Square from another; ſo likewiſe it helps us to ſome Foundations whereon, among the reſt, the ſtructure of the Tables of Sines relies, &c. Whoſe uſe we have already partly ſhewn in *Schol.* 1 and 4. 1. If ſeveral Squares are to be collected into one Sum, having joined the Sides of two of them ſo as to form a Right Angle, *e.g.* A*B* and BC (*Fig.* 68. N° 1.) the Hypothenuſe AC being drawn, is the Side of a Square equal to them both; and if this Hypothenuſe AC be removed from B to D, and the Side of the third Square from B to E, the new Hypothenuſe DE will be the Side of a Square equal to the three former taken together. 2. If the Square of the ſide MN (N° 2.) is to be ſubtracted from the Square of the ſide LM. Having deſcribed a Semicircle upon LM, and placed the other MN within that Semicircle, then draw the Line LN and that will be the Side of the remaining Square. 3. Having the Right Sine EG of any Arch ED given (but how to find the Primary Sines we will ſhew in another place) you may obtain the Sine Complement CG or EF, by the

preceding

Numb. viz. by ſubtracting the □ of the given Sine from the □ of the Radius; and moreover the verſed Sine GD by ſubtracting the Sine Complement CG from the Radius CD. 4. The Squares of the verſed Sine GD, and of the Right ſine EG being added together, give the □ of the Chord ED of the ſame Arch, (which all are evident from the *Pythagorick Theorem*) and half of that EH gives the Right Sine of half that Arch. 5. From the Right Sine EG you have the Tangent of that Arch, if you make, as the Sine Complement CG to the Right Sine GE, ſo the whole Sine CD to the Tangent GI. 6. Laſtly, From theſe *Data* you may alſo have the Secants (if required) thus, as the Sine Complement CG to the W. S. CE, ſo the W. S. CD to the Secant CI; or as the Right Sine EG to the W. S. E.C. ſo the Tangent ID to the Secant IC; both which are evident by our 34th Propoſition.

Conſect. 9. If the Quadrant of a Circle (CBEG, Fig. 70.) be inclined to another Quadrant (CADG) and two other Perpendicular Quadrants cut both of them, *viz.* FBAG and FEDG, and the latter do ſo in the extremities of them both) having let fall Perpendiculars from the common Sections E and B, thro' the Planes of the Perpendicular Quadrants, and the inclined Quadrant, (*viz.* on the one ſide EG and BH, as Right Sines of the Segments EC and BC; on the other EI and BK, as Right Sines of the Segments ED and BA) you'l have 2 Triangles EIG and BKH Right Angled at I and K, Equiangular at G and H (by reaſon of the ſame inclination of the Plane CBEGC) and conſequently ſimilar, by our 34th Propoſition; wherefore as the Sine EG to the Sine EI, ſo the Sine BH to the ſine BK, or as EG to BH ſo EI to BK, and contrariwiſe.

SCHOLIUM VI.

Hence you have ſeveral Rules of *Spherical Trigonometry* for reſolving Right Angled △△ (α) 1. Having given in the Rightangled △ ABC the Hypothenuſe BC and the Oblique Angle ACB, for the Leg AB oppoſite to this Angle, make: as the ſine T (EG) to the ſine of the Hypoth. (BH) ſo the ſine of the given Angle (EI) to the ſine of the Leg ſought (BK). 2. Having given the Hypothenuſe

(α) *Lansberg. Geom. Triang. lib.* 4. *Prop.* 12.

nuſe BC and the Leg AB for the oppoſite Angle ACB make as the ſine Hypoth. (BH) to *S*. T. (EC) ſo the ſine of the given Leg (*B*K) to the ſine of the Angle ſought (EI) 3. Having given the ſide AB and the Angle oppoſite to it ACB, for the Hypothenuſe BC (ſuppoſing you know whether it be greater or leſs than a Quadrant) make as the ſine of the given Angle EI to the ſine T. (EG) ſo the ſine of the given Leg (*B*K) to the ſine of the Hypoth. BH). 4. Having given in the Right Angled △ EBF (which we take inſtead of ABC, that ſo we may not be obliged to change the Figure) one Leg EB and the Hypothenuſe BF for the other Leg *E*F, you may find its complement, if you make as the ſine Complement of the given ſide (BH) to *S*. T. (EG) ſo the ſine Complement of the Hypothenuſe (BK) to the ſine Compl. of the ſide ſought (EI) 5. Having both Legs EB and EF given, for the Hypothenuſe BF its Compl BA may be found thus: as *S*. T. (EG) is to the ſine Cpmpl. (BH) of one ſide EB, ſo the ſine Compl. (EI) of the other ſide (EF) to (BK) the ſine Compl. of the Hypothenuſe.

6. Having given in the ſame Right Angled Triangle EBF one Leg EF, and the Angle adjacent to it ETB, firſt prolong into whole Quadrants BA to *f*, that A *f* may = BF Hypoth. & BC to *e* that C*e* may = EB, and AC to *d* that C*d* may = DA the meaſure of the given Angle EFB: ſecondly from *d* thro' *e* and *f* let fall a Quadrant thro' the extremities of the Quadrants B*f* and B*e*, that ſo the △ C *de* may be Right Angled, in which there are given the Hypoth. C*d* = to the given Angle, and the Angles C = to the Compl. of the given Leg (*viz*. to the Arch ED) and ſo, thirdly, there is ſought the ſide *d e*, as the Complement of the Arch *e f*, or of the Angle ſought ABC, or EBF; *viz*. by the firſt caſe of this, by inferring, as *S*. T. to the ſine Hypoth. *c d* (*i.e.* of the given Angle EFB;) ſo the Angle *dce* (*i. e.* DE the Compl. of the given Leg RF) to the ſine *d e* (as the Compl. of the Angle *f*B*e* or EBF.)

7. Having given, in the ſame Triangle, the ſide EF and the oppoſite Angle EBF (*i. e.* the Arch *ef*) for the other Angle EFB, (that is the Hypoth. *cd* in the △ *cde*) make by the third of this:

As the *Sine* of the Angle *d c e* (*i. e.* the ſine Compl. of the given Leg DE) to the *S*. T. ſo the ſine of the Leg *d e* (*i. e.* the ſine Compl. of the Angle EBF) to the Hypoth. *cd* (*i.e.* the ſine of the Arch DA or Angle EFB.).

8. Having the Oblique Angles given to find either of the ſides, *viz.* EF; which may be done thus by the ſecond of this:

As the ſine of the Hypoth. *cd* (*i. e.* the ſine of the Angle at F) to the W. S. ſo the ſine *de* (*i. e.* the ſine Compl. of the Angle at B) to the ſine of the Angle *dce* (*i.e.* the ſine Compl. of the ſide ſought EF.)

Conſect. 10. The ſame being given as in *Conſect.* 7. if inſtead of the Right Sines EI and BK, you erect Perpendicularly DL and AM (*Fig.* 71) becauſe of the ſimilitude of the Triangles DGL and AHM, you'l have, as DG ſine T. to DL the Tangent of the Arch DE, ſo AH the Right Sine of the Arch AC to AM the Tangent of the Arch AB; or as DG to AH, ſo DL to AM, and contrariwiſe.

SCHOLIUM VII.

HEnce flow the other Rules of *Spherical Trigonometry* for Reſolving Right Angled Triangles, *viz.* 9. Having given the ſide AC in the △ABC, and the adjacent Angle ACB, for the other ſide AB, make as the W. S. (DG) to the ſine of the given ſide (AH) ſo the Tangent of the given Angle (ACB) to the Tangent of the Angle ſought (AB.) 10. Having given the ſide (AB) and the oppoſite Angle (at C) for the other ſide (AC, ſo you know whether it be greater or leſs than a Quadrant) make as the Tangent of the given Angle (DL) to the Tangent of the given Leg (AB) ſo the whole S. (DG) to the ſine of the Leg ſought (*viz.* at AH.) 11. Both ſides being given, for the Angles, make, as the ſine of one Leg (AH) to the W. S. (DG) ſo the T. of the other Leg (AM) to the Tangent of the Angle oppoſite to the ſame (at C.) 12. Having given moreover in the Right Angled Triangle EBF the Hypotheuſe (BF) and the Angle (EFB) for the adjacent ſide EF, make, as the ſine Compl. of the given Angle (AH) to the W. S. ſo the Tangent Compl. of the Hypoth. (AM) to the Tang. Compl. of the Leg ſought (DL) 13. Having given the ſide (EF) and the adjacent Angle F for the Hypoth. BF make; as the W. S. to the ſine Compl. of the given Angle (AH) ſo the Tangent Complement of the given Leg (DL) to the Tangent Compl. of the Hypoth. (AM.) 14. Having given the Hypoth. (BF) and one ſide EF for the adjacent Angle (F) make as the Tang. Compl. of the given Leg (D.L.) to the W. S. ſo the Tang. Compl. of the Hypoth. (AM) to the Sine Compl. of the Angle ſought (AH). 15. Having given the Hypoth. (BF, *i. e.* the arch Af, or the angle at *d*) and either of the oblique angles (at F) for the other angle (EBF) make by help of the new Triangle *c d e*, by the 12*th* of this.

As the Sine Compl. of the angle *c d e* (*i. e.* the Sine Compl. of the Hypoth. (AH) to the W. S. ſo the Tang. Compl. of the Hypoth. *c d* (*i. e.* Tang. Compl. of the given angle) to the Tang. Compl. of the Side *d e* (*i. e.* to the Tang. of the angle ſought ABC or EBF.)

16. Having given the oblique Angles to find the Hypoth. (BF, or the arch A*f*, or the angle *c d e*) it is done by the 14 of this *Schol.*

As the Tang. Compl. of the Leg *d e* (*i. e.* the Tangent of the angle ABC or EBF) to the W. S. ſo the Tangent Complement of the Hypoth. *c d* (*i. e.* the Tangent Complement of the other angle EFB) to the Sine Compl. of the angle *c d e* (*i.e.* the Sine Compl. of the Hypoth. BC ſought.)

So that now we have with *Lanſbergius* (but much more compendiouſly) Scientifically Reſolved all the Caſes of Rigt-angled Triangles; the Reſolution of Oblique-angled ones only now remaining.

Conſect. 11. In Oblique-angled Spherical Triangles, as well as Right-angled ones, the Sines of the angles are directly proportional to the Sines of the oppoſite Sines. 1. Of the Right-angled ones this is evident from N° 3. *Schol.* 6. and from the 9th *Conſect.* For as the Sine of the angle A (*Fig.* 72.) to the Sine of BD, ſo the W. S. (*i.e.* of the angle D) to the Sine of AB. 2. The ſame is immediately evident of an Oblique-angled Triangle ABC, reſolved into 2 Right-angled ones. For,

The Sine of the angle C is to the Sine of BD as the ſine of the gle D to the Sine of AB; and alſo,

The Sine of the angle C to the Sine of BD as the Sine of the angle D to the ſine of BC, by the 1.

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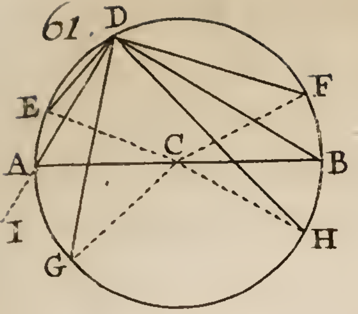

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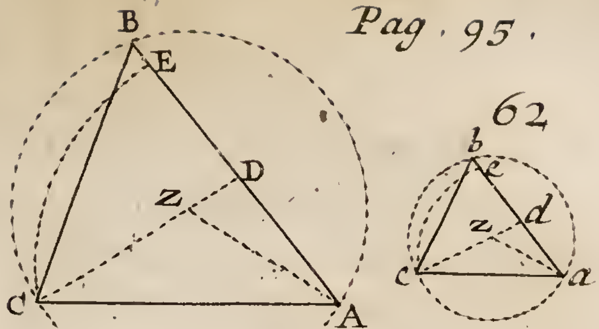

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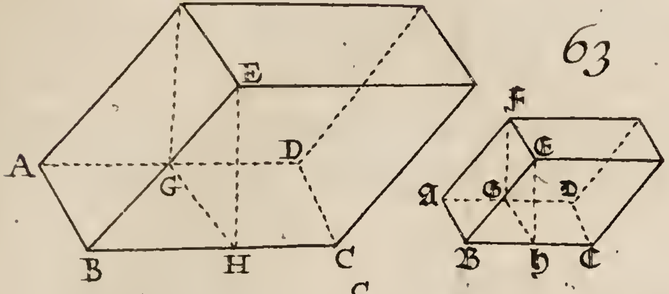

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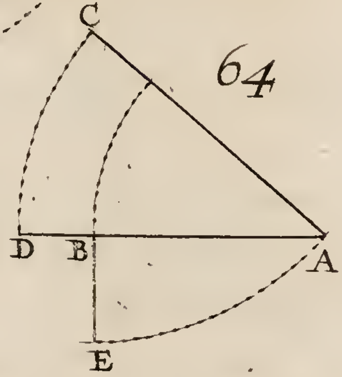

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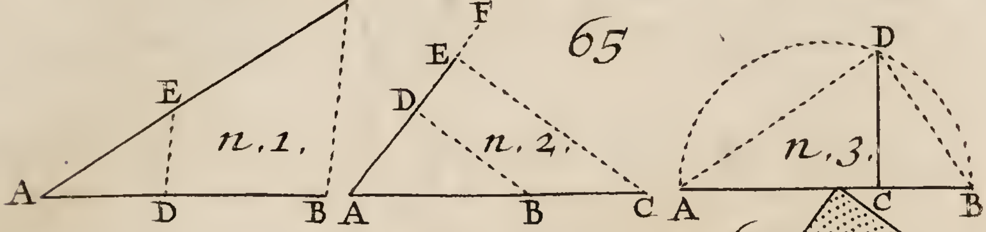

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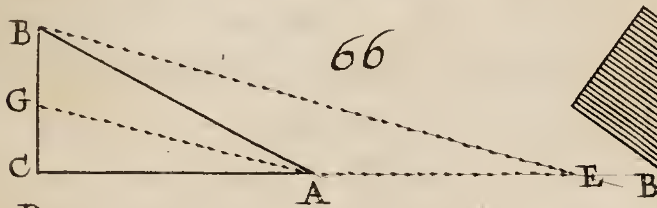

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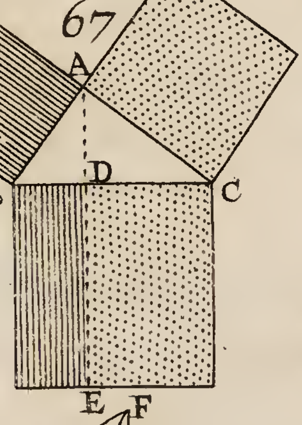

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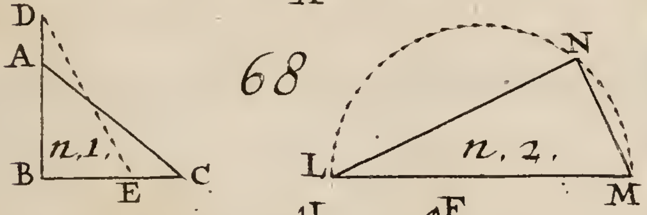

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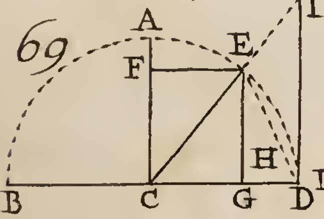

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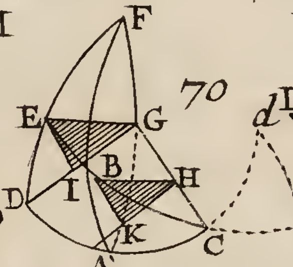

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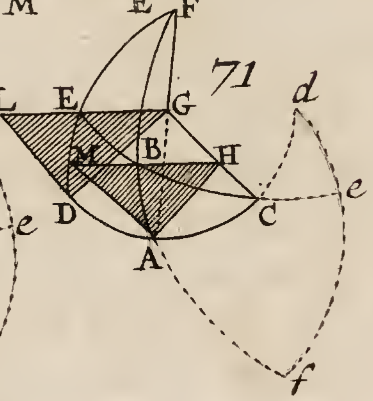

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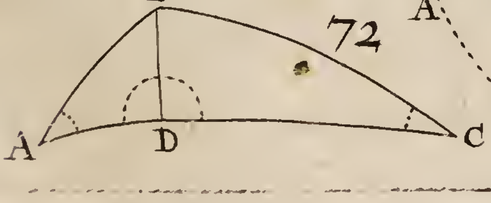


In each Proportionality the means are the Sines of BD and D; therefore the Rectangles of the Extremes of the Sines of AB into the Sine of A, and the Sine of BC into the Sine of C, will be equal among themſelves, ſince the Rectangles of the ſame Means are equal, by *Prop.* 18. therefore by *Prop.* 19. as the Sine of A to the Sine of BC, ſo the Sine of C to the Sine of AB, Q.E.D.

SCHOLIUM VIII.

THe latter may appear of Oblique-angled Triangles after this way alſo; ſince the Sine of the angle A is to the Sine of BD as the Sine of the angle D to the ſine of AB, call the firſt a, the ſecond $e\,a$, the third b, the fourth $e\,b$; and becauſe the Sine of the angle C (which we call c) is likewiſe to the Sine of BD (*i. e.* to $e\,a$) as the Sine of D (*i. e.* b) to the Sine of BC (which will conſequently be $\frac{e\,a\,b}{c}$) it will be manifeſt, that the Sine of the angle

A is to the ſine of BC as the ſine of the angle C to the ſine of *AB*,
i.e. as a to . . $\frac{eab}{c}$, *i. e.* . . c to . . eb.

by multiplying the Means and Extremes, whoſe Rectangles are on both ſides $e\,a\,b$. Therefore as by the preſent and precedent *Conſectary* 7, it is univerſally true, *That in any Triangle whether Right Lined or Spherical, Right-Angled or Oblique-angled, the Sides or their Sines, are to one another, as the Sines of their oppoſite Angles* (which therefore is commonly called a *Common Theorem*:) ſo alſo hence flow 2 new Rules of *Spherical Trigonometry* for Oblique-angled Triangles, like thoſe we found in *Schol.* 4.

To find		I. *The other Angles.*
From 2 ſides given of an angle oppoſite to one of them,	by inferring	As the ſine of the ſide oppoſite to the given angle to the ſine of the other ſide, ſo the ſine of the given angle to the ſine of the angle ſought.

II. *The*

II. *The other Sides.*

From one ſide and the angles given,	by inferring	As the ſine of the angle oppoſite to the given ſide to the ſine of that ſide, ſo the ſine of the angle oppoſite to the ſide ſought to the ſine of the ſide ſought.

And thus we have reduced all the Caſes and Rules of Spherical Trigonometry to their original Fountains (for from 2 Sides given and the interjacent Angle, or 2 Angles and their adjacent ſide, we may find the reſt in Oblique-angled Triangles, by reſolving them into 2 Right-angled ones; and ſo by the Rules we have deduc'd in *Schol.* 6 and 7) excepting two Caſes, *viz.* when from 3 ſides given, the Angles, or from 3 Angles the Sides are ſought; to reſolve which, we are ſupplied with Rules from the following

Conſ. 12. In the given Oblique-angled Spherical Triangle ABC (*Fig.* 73.) whoſe Sides are unequal and each leſs than a Quadrant, having produced the ſides AB and AC to the Quadrant AD and AE, and effected beſides what the Figure directs, then will

The Arch DE be the Meaſure of the angle A, AF=AC, and ſo FB the difference of the Sides AB and AC.

BC=BG, and ſo GF the difference of the third ſide, and the differences of the reſt FB.

But now, 1. As EH or DH to CM or FM ſo will PH be to NM (by reaſon of the Equiangular Triangles EPH and CNM;) therefore by *Prop.* 26. ſo will alſo D P be to F N. Make therefore $DH = a$ $FM = ea$, $DP = b$ $FN = eb$.

AI the R. Sine of the ſide AB.
CM the Sine of the Side AC.
GL the Sine of G*B*, or of the ſide BC.
FK the R. Sine of the Arch F*B*.
*B*I the verſed Sine of A*B*.
*B*L the verſ. Sine of G*B* or *B*C.
*B*K the verſed *S*ine of F*B*.
KL or NO the difference of the verſed Sines we have now mentioned.
EP the Right *S*ine and DP the verſed *S*ine of the arch DE.
CN the R. *S*ine and FN the verſed *S*ine of the arch FG.

2. By reaſon of the Equiangular ΔΔ FNO and HAI (for FNO is Equiangular to the Δ FKQ, and that to the Δ HQM, by reaſon of the Vertical Angles at Q; and that alſo to the Δ HAI by reaſon of the Common Angle at H) and you have alſo,

As HA,
Or DH to AI, ſo FN to NO.
a — oa — eb — oeb.

Wherefore now, 3. you'l have evidently,
the □ DH to FM into AI as DP to NO.

aa —— oeaa —— b —— oeb.
i. e. a —— oea —— b —— oeb.
DH NO DP.

And Inverſly as *oea* to *a* ſo *oeb* to *b.*

SCHOLIUM IX.

Since therefore the Radius DH or *a* is known, and alſo NO the Difference of the verſed Sines *B*L and *B*K, it is evident, that DP the verſed Sine of the angle A will be known alſo; ſuppoſing that the firſt Quantity *oea* is likewiſe known. *But* this may be had by another Antecedent Inference, if you make,

as AH to FM ſo AI to a fourth *oea.*
a ea oa

Hence therefore ariſes, 1. the Rule: Having given the 3 Sides of an Oblique-angled Triangle, to find any one of the Angles, *viz.* by inferring,

1. As the Sine of T to the Sine of R, one of the ſides comprehending AC; ſo the ſine of the other ſide AB to a fourth,

DH or AH — FM — AI — *oea.*
a — ea — oa.

2. As this fourth to the ſine of T, ſo the difference of the verſed ſines of the third ſide, BC, and the differences of the others to the verſed ſine of the Angle ſought, *viz.*

oea — a — NO — DP.
oeb b.

But ſince the ſides of a Spherical Triangle may be changed into Angles, and contrariwiſe the ſides being continued [as if the ſide AB of the given Triangle ABC (*Fig.* 47.) be continued a Circle, the reſt into Semicircles from the Poles *b* and *c*, and likewiſe the *S*emicircle HI from the Pole A, and the *S*emicircle FG from the Pole B, and the *S*emicircle EA from the Pole C, you'l have a new Triangle *a*, *b*, *c*, the 3 angles of which will be equal to the 3 ſides of the former ABC; as the angle *a* or its meaſure IG, is equal to the ſide AB, by reaſon each makes a Quadrant joined with the third arch AG; but the meaſure of the angle *b*, is the ſide AC (*viz.* in this caſe wherein the ſide AC is a Quadrant, in the other wherein it would be greater or leſs than a Quadrant, it would be the meaſure of the angle of the Compl. for then the *S*emicircle H*ab* deſcribed from the Pole A, would not paſs thro' C but beyond or on one ſide of C. See *Pitiſc. lib.* 1. *Prop.* 61. *p. m.* 25.) ——— the angle *c* or its meaſure KL, is equal to the ſide BC, becauſe with the third KC they make the Quadrants BK and CL] Therefore, 2. Having given the three Angles of the Oblique-angled Triangle *a b c*, you may find any ſide, *e. g.* *a c*, if there be ſought the Angle ABC, or rather its Complement KBF, or its meaſure FK = *a c*, ——— from the 3 ſides given of the △ ABC, by the preceding Rule, by inferring, *viz.* 1. As *S.* T. to the ſine R, of one ſide comprehending the angle of one ſide AB (*i.e.* of one angle *a* adjacent to the ſide ſought) ſo the ſine of the other ſide BC (*i. e.* of the other angle C) to a fourth.

2. As the fourth to the *S.* T. ſo the difference of the verſed *S*ines of the third ſide AC, and the differences of the others (*i. e.* of the 3d angle *b*, and the differences of the reſt) to the verſed *S*ine of the comprehended angle, or Complement to a *S*emicircle (*i. e.* of the ſide ſought *a c*.)

Propoſition XXXV.

SImilar Plane Figures (α) *are to one another in Duplicate Proportion of their Homologous Sides.*

Demonſtration.

For, 1. the Baſes of 2 ſimilar Triangles or Parallelograms (for which any 2 Homologous Sides, *e. g.* AB and 𝔄𝔅 (*Fig.* 75.) may be taken) and Perpendiculars let fall thereon DE and 𝔇𝔈, will be by *Conſect.* 1. *Prop.* 34, as *a* to *e a*, *b* to *e b*. Therefore the Parallelograms and Triangles themſelves, will be as *ba* to *eeba*, by *Conſect.* 7 and 8. *Def.* 12. *i. e.* by *Def.* 34. in duplicate Reaſon of their Perpendiculars or aſſumed Sides, which is moſt conſpicuous in Squares, which putting *a* for the ſide of one, and *e a* for the other, are to one another as *a a* to *e e a a*.

2. Like Polygons are reſolved into like Triangles, when the Triangles ABC and 𝔄𝔅ℭ, and alſo AED and 𝔄𝔈𝔇 are Equiangular, by *Conſect.* 3. *Prop.* 34. but CAD and ℭ𝔄𝔇, are alſo Equiangular, becauſe each of their angles are the remainder of equal ones, after equal ones are taken from them. Wherefore the firſt Triangles are in duplicate Proportion of the ſides BC and 𝔅ℭ; the ſecond likewiſe of the ſides CD and ℭ𝔇; the third are alſo in the ſame Proportion of the ſides DE and 𝔇𝔈, *&c. i. e.* (ſince by the Hypoth. BC has the ſame reaſon to 𝔅ℭ as CD to ℭ𝔇, and DE to 𝔇𝔈) each to each is in duplicate Proportion of the ſides BC to 𝔅ℭ, or CD to ℭ𝔇, by the firſt of this: Therefore by a *Syllepſis*, the whole Polygons are in duplicate Proportion of the ſame *Sides*: Which is the ſecond thing to be demonſtrated.

(α) *Eucl.* 19 & 20 *lib.* 6.

3. Circles and their like *Sectors*, are as the Squares of their Diameters, by *Prop.* 32. therefore in duplicate Proportion of them, by the firſt of this: Which is the third thing: Therefore ſimilar Plane Figures, *&c.* Q. E. D.

CONSECTARYS.

I. THerefore 2 ſimilar Plane Figures are one to another, as the firſt Homologous *Side*, to a third Proportional, by vertue of *Definition* 34.

II. Any two Figures deſcribed on 4 Proportional Lines (α) and ſimilar to 2 others, are likewiſe Proportional, and contrariwiſe; for if the ſimple Reaſons or Proportions of Lines be the ſame, their duplicate Proportions will be the ſame alſo, and reciprocally.

(α) *Eucl. prop.* 22. *lib.* 6.

SCHO-

SCHOLIUM.

BUT as this ſecond Conſectary confirms Prop. 22 and its Scholium, ſo the firſt teaches us a twofold Geometrical Praxis. 1. A Way to expreſs the Proportion of ſimilar Figures by two Right Lines, *viz.* by finding a third Proportional to their Homologous Sides. For as the ſide of the firſt to this third, ſo will be the firſt Figure to the ſecond. 2. A way to augment or diminiſh any given Figure in a given Reaſon or Proportion, *viz.* by finding a mean Proportional between any ſide of the given Figure, and another Line which ſhall be to that in a given Proportion, and then by deſcribing thereon a ſimilar or like Figure.

Propoſition XXXVI.

SImilar or like Solid Figures, are to one another in triplicate Proportion of their Homologous Sides.

Demonſtration.

For, 1. The ſimilar Baſes of two ſimilar Parallelepipedons (and conſequently alſo of Priſms and Cylinders, by *Conſect.* 4 and 5. *Definit.* 16. and alſo of Pyramids and Cones, *by Conſect.* 3 and 4, of *Definit.* 17.) are, as *a b* to *e e a b*, by (*a*) the preceding Propoſition, and their Altitudes as *c* to *e c*, by *Conſect.* 2. *Prop.* 34.

Therefore Parallelepipeds, Cylinders and Priſms (and ſo the third part of theſe, Cones and Pyramids) will be as *abc* to e^3abc, by *Conſect.* 3, 4, 5. *Definit.* 16. *i. e.* they will be, by *Definit.* 34. & *Conſect.* 1 and 2. *Prop.* 34. in Triplicate Proportion of their Perpendiculars or Homologous Sides. Which is eſpecially Conſpicuous in Cubes; which, putting *a* for the Side of one, and *e a* for the ſide, are to one another as a^3 to e^3a^3.

2. Polyedrous or many ſided Figures, may be reſolved into Pyramids of ſimilar Baſes and Altitudes; which is evident of Regular ones, from the *Conſect.* of *Definit.* 21. and cannot be difficult to underſtand alſo of Irregular ones; becauſe the like

inclination of their Planes every where ſimilar and equal in number, neceſſarily require that the whole Altitudes of ſimilar Polyedrous Solids, as well as ſimilar Parallelepipedons, by *Conſect.* 2. *Prop.* 34. ſhould be in ſubduplicate Proportion of their Baſes, and ſo theſe being likewiſe divided in C and ℭ (*Fig.* 76. N° 1.) the parts of their heigths GC and 𝔊ℭ, will be in the ſame Proportion: Whence, *e. g.* 2 Pyramids ſtanding on ſimilar Baſes ABDEF and 𝔄𝔅𝔇𝔈𝔉, and having like Altitudes GC and 𝔊ℭ, will neceſſarily be like or ſimilar; and the ſame thing may be likewiſe judged of others.

Or yet, to ſhew it more evidently, the Polyedrous Solids may be reſolved into like Triangular Priſms; for, *e.g.* each of the Triangles of their ſimilar Baſes ABDEF and 𝔄𝔅𝔇𝔈𝔉 (N° 2.) are ſimilar, *viz.* 𝔞 𝔟 𝔣 and *a b f*, 𝔄𝔅𝔉 and ABF, by the preceding *Prop.* N° 2. The Planes 𝔄𝔅𝔟𝔞 and AB*ba*, alſo 𝔄𝔞𝔣𝔉 and A *a f* F, are ſimilar by the Hypoth. and conſequently alſo the Planes 𝔅𝔟𝔣𝔉 and B*bf*F (𝔣𝔟 is to 𝔟𝔞 as *fb* to *ba*, and alſo in the one 𝔟𝔞 to 𝔟𝔅, as *ba* to *b*B in the other; therefore *ex equo* as 𝔣𝔟 to 𝔟𝔅 ſo *f b* to *b*B, *&c.*) and ſo the whole Triangular Priſms will be ſimilar, by *Definit.* 35. and ſo of others. Therefore ſimilar Polyedrous Solids will be in the ſame Proportion as ſimilar Pyramids, or Triangular Priſms, *i. e.* by the firſt of this in Triplicate Proportion of their Sides.

3. Spheres are as the Cubes of their Diameters (β) by *Conſect.* 3. *Prop.* 32. Therefore they are by the firſt of this, as a^3 to $e^3 a^3$. Therefore ſimilar or like Solids are in Triplicate Proportion of their Homologous Sides. Q. E. D.

(α) *Eucl. prop.* 12. *lib.* 12. *of Cones and Cylinders.*

(β) *Eucl.* 18. *lib.* 12.

CHAP. VI.

Of the Proportions of Magnitudes of divers ſorts com pared together.

Propoſition XXXVII.

THE *Parallelogram ABCD* (Fig. 77. N. 1.) *is to the Triang BCD upon the ſame baſe DC, and of the ſame heighth as* 2 1. This has been already Demonſtrated in *Conſect.* 3. *Def nit.* 12. Here we ſhall give you another

Demonſtration.

Suppoſe, 1. the whole Baſe CD divided into four equ parts by the tranſverſe Parallel Lines EG, HK, LN, then wi (by reaſon of the ſimilitude of the ΔΔ DGF, DKI, DNM DCB.) GF be 1, KI 2, NM 3, CB 4; and having furthe more continually Biſected the Parts of the Baſe, the Indiviſibl or the Portions of the Lines drawn tranſverſly thro' the Tria gle will be 1, 2, 3, 4, 5, 6, 7, 8, &c. *ad infinitum*, all long in an Arithmetical Progreſſion, beginning from the Poi D, as 0; to which the like number of Indiviſibles always a ſwer in the Parallelogram equal to the greateſt, *viz.* the Li BC. Wherefore by the 4th *Conſect.* of *Prop.* 16. all the Ind viſibles of the Triangle, to all thoſe of the Parallelogram take together, *i. e.* the Triangle it ſelf to the Parallelogram, is as to 2. Q. E. D.

SCHOLIUM.

NOW if any one ſhould doubt whether the Triangle Parallelogram may be rightly ſaid to conſiſt of an in nite number of Indiviſible Lines, he may, with Dr. *Wall*

inſtea

inſtead of Lines, conceive infinitely little Parallelograms of the ſame infinitely little Heighth, and it will do as well. For having cut the Baſe (N° 2) into 4 equal parts by tranſverſe Parallels, there will be circumſcribed about the Triangle ſo many Parallelograms of equal heighth, being in the ſame Proportion as their Baſes, by *Prop.* 28. *i. e.* increaſing in Arithmetical Progreſſion. In the following Biſection, there will ariſe 8 ſuch Parallelograms approaching nearer to the Triangle, in the next 16, &c. ſo that at length infinite ſuch Parallelograms of infinitely leſs heighth, and ending in the Triangle itſelf, will conſtitute or make an infinite Series of Arithmetical Proportionals, beginning not from 0 but 1; to which there will anſwer in the Parallelogram infinite little Parallelograms of the ſame heighth, equal to the greateſt. Whence it again follows, by *Conſect.* 9. *Prop.* 21. that the one Series is to the other, *i. e.* the Triangle to the Parallelogram as 1 to 2; which being here thus once explained, may be the more eaſily applied to Caſes of the like nature hereafter.

CONSECTARYS.

I. Since in like manner in the Circle (*Fig.* 79.) the Peripherys at equal intervals from one another, as ſo many Elements of the Circle, increaſe in Arithmetical Progreſſion; the Sum of theſe Elements, *i. e.* the Circle it ſelf will be to the Sum of as many Terms equal to the greateſt Periphery, *i. e.* to a Cylindrical Surface, whoſe Baſe is the greateſt Periphery, and its Altitude the Semidiameter, as 1 to 2.

II. Hence the Curve Surface of a Cylinder circumſcribed about a Sphere, *i. e.* whoſe Altitude is equal to the Diameter, is Quadruple to its Baſe.

III. Alſo the Sector of the Circle *bac*, to a Cylindrical Surface, whoſe Baſe is the Arch *bc*, but its Altitude the Semidiameter *ac*, is as 1 to 2.

IV. And becauſe the Surface of the Cone BCD is to its circular Baſe, as BC to CA, *i. e.* as the $\sqrt{2}$ to 1, by *Schol. Prop.* 17. the Cylindrical Surface, the Conical Surface and the Circular one we have hitherto made uſe of will be as 2, $\sqrt{2}$ and 1, and conſequently continually Proportional.

SCHOLIUM

ALL which may alſo abundantly appear this way, *viz.* by putting for the Diameter of the Circle *a*, for the Semidiameter $\frac{1}{2}a$ and for the Circumference *ea*, you'l have the Area of the Circle $\frac{1}{4}eaa$ by *Conſect.* 1 *Definit.* 31. and Multiplying the heighth of the Cylinder AB *i. e.* $\frac{1}{2}a$ by the Periphry *e a* you'l have the Cylindrical Surface $\frac{1}{2}eaa$ by *Conſect.* 6. *Definit.* 18. as now is evident alſo by *Conſect.* 1, 2 and 3. Now if you would alſo have the Surface of the Cone, ſince it's ſide by the *Pythagorick Theorem* is $\sqrt{\frac{1}{2}aa}$ and the half of that $\frac{1}{2}\sqrt{\frac{1}{2}aa}$ *i. e.* (by N° 2 of *Schol. Prop.* 22.) $\sqrt{\frac{1}{8}aa}$; and this half being multiplied by the Periphery of the Baſe *ea*, you'l have (by virtue of *Conſect.* 4. *Definit.* 18.) the Surface of the Cone $ea\sqrt{\frac{1}{8}aa}$ *i. e.* (by the *Schol.* juſt now cited) $\sqrt{\frac{1}{8}eeaa^4}$: So that now appears alſo the 4*th Conſect.* of this; becauſe the Rectangle of thoſe Extremes $\frac{1}{2}eaa$ and $\frac{1}{4}eaa$ is $\frac{1}{8}eeaa^4$ as well as the ſquare of the mean.

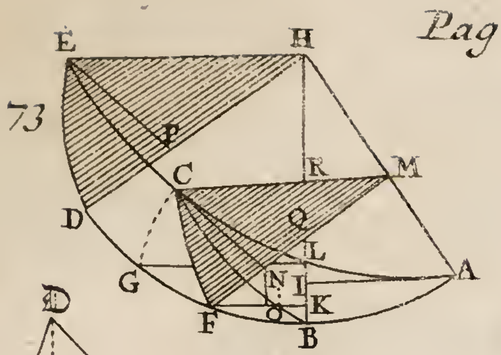
Propoſition XXXVIII.

A *Parallelepiped* (α) *BF* (Fig. 77. N° 3.) *is to a Pyramid ABCDE upon the ſame Baſe BD and of the ſame heighth, as* 3 *to* 1. This was Demonſtrated in *Conſect.* 3. *Definit.* 17. but here we ſhall give you another.

Demonſtration.

Suppoſe 1 the whole Altitude BE divided into 3 equal Parts, by tranſverſe Plains Parallel to the Baſe, then will (by reaſon of the Similitude of the Pyramids abcd E. $\mathfrak{ACBDE}$ and ABCDE) the Baſes abcd, $\mathfrak{ABCD}$ and ABCD be by *Conſect.* 2. *Prop.* 34. and *Conſect.* 3. *Definit.* 17 in duplicate Proportion of the Altitudes, *i. e.* in duplicate Arithmetical Progreſſion 1, 4, 9, moreover 2, biſecting the parts of the Altitude, the quadrangular Sections now double in Number (as the Indiviſibles or Elements of the propoſed Pyramid) will be as 1, 4, 9, 16, 25, 36,

(α) *Euclid. Prop* 7 *Coroll. lib.* 12.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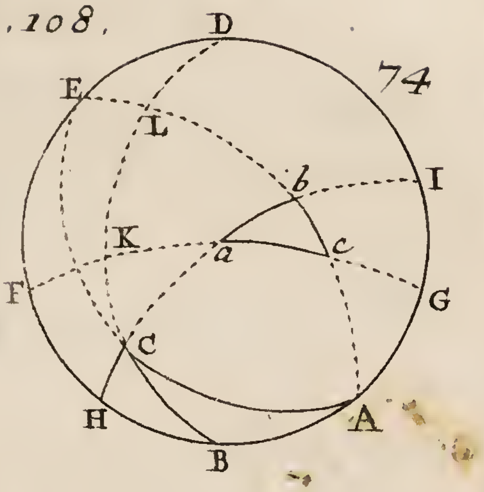
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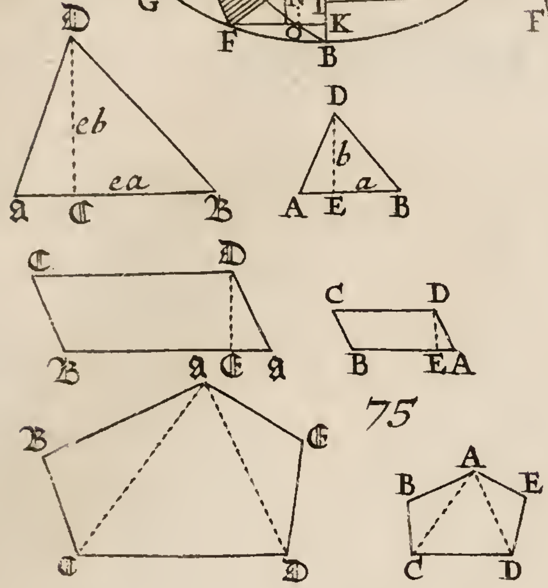
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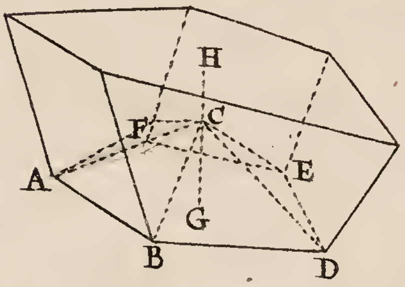

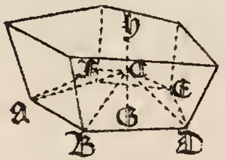
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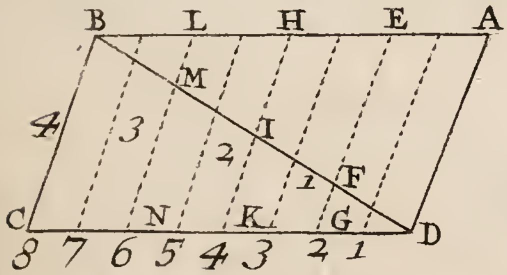

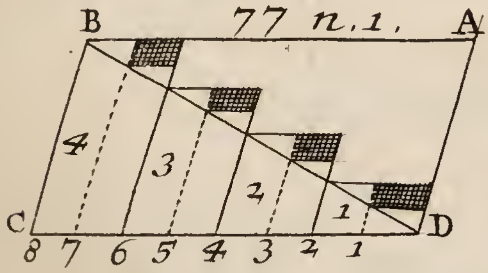
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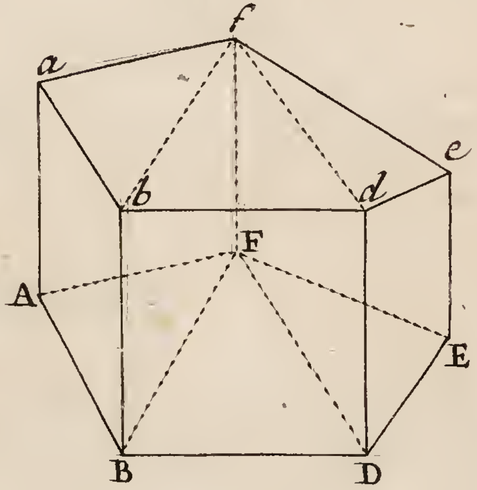

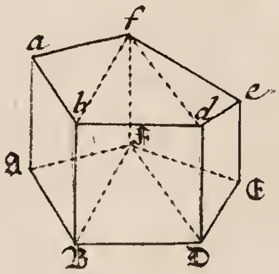
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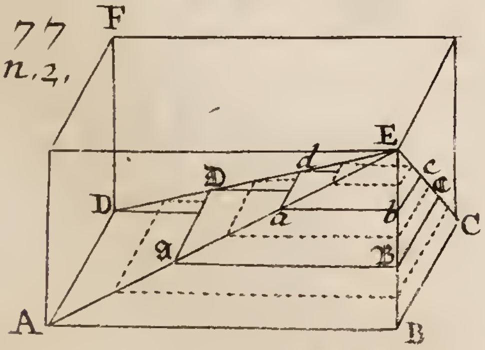

77 n.2.

36, *&c. ad Infinitum*, all along in a duplicate Arithmetical Proportion; while in the mean time there anſwer to them as many Elements in the Parallelepiped equal to the greateſt ABCD, wherefore by *Conſect.* 10. *Prop.* 21. all the Indiviſibles of the Pyramid taken together will be to all the Indiviſibles of the Parallelepiped alſo taken together, *i. e.* the Pyramid it ſelf to the Parallelepiped, as 1 to 3. Q. E. D.

CONSECTARY.

THis Demonſtration may be eaſily accommodated to all other Pyramids and Priſms, and alſo Cones and Cylinders, (α) ſince here alſo (*Fig.* 78.) the circular Planes *ba*, 𝔅𝔄, and BA are as the ſquares of the Diameters, and ſo as 1, 4, 9. and ſo likewiſe all the other Elements of the Cone by continual biſection are in duplicate Arithmetical Progreſſion; when in the mean time there anſwer to them in the Cylinder as many Elements equal to the greateſt BA, *&c.*

(α) *Euclid. Prop.* 10. *lib.* 12.

Propoſition XXXIX.

A *Cylinder is to a Sphere inſcribed in it* i. e. *of the ſame Baſe and Altitude as* 3 *to* 2.

Demonſtration.

Suppoſe 1 (*Fig.* 80.) the half Altitude GH (for the ſame proportion which will hold when demonſtrated of the half Cylinder AK and Hemiſphere AGB, will alſo hold the ſame of the whole Cylinder to the whole *S*phere) to be divided into 3 equal parts, then will AH, C1, E2, be mean proportionals between the *S*egments of the Diameter by *Prop.* 34. *Schol.* 2. N° 3, and ſo by *Prop.* 17. the Rectangles LHG, L1G, L2G equal to the Squares AH, C1, E2, being in order as 9, 8 and 5, and alſo 2*dly*, having biſected the former parts of the heighth, the ſix Squares cutting the Sphere Croſs ways will be found to be as 36, 35, 32, 27, 20, 11. *&c.* in the progreſſion we have ſhewn at large in *Conſect.* 12. *Prop.* 21. Wherefore ſince all the Indiviſibles of the Hemiſphere, *viz.* the circular Planes anſwering to the *S*quares of the ſaid tranſverſe

Diameters have the ſame proportion of Progreſſion, by *Prop.* 32. and there anſwer to them the like number of Elements in the Cylinder equal to the greateſt AH: All theſe theſe taken together will be to all the other taken together *i. e*, the whole Cylinder AK to the whole Hemiſphere AGB by vertue of the aforeſaid *Conſect.* 12. as 3 to 2 (α) Q. E. D.

(α) *Archim.* 32 (*al.* 31.) *lib.* 1 *de Sphær.*

SCHOLIUM. I.

HOnoratus *Fabri* elegantly deduces this *Prop. a priori*, in a genetick Method in his *Synopſis Geom.* p. 318. (which alſo *Carolus Renaldinus* performs from the ſame common Foundation in *lib.* 1. *de Compoſ.* and *Reſol.* p. 301, and the following, but after a more obſcure way and from a demonſtration further fetch'd) *Fabri's* is after this Method: The whole Figure (81) AL being turned round about BZ, the Quadrant ADLBA will deſcribe an Hemiſphere, the Square AZ a Cylinder and the triangle BML a Cone all of the ſame Baſe and Altitude. Since therefore Circles are as the ſquares of their Diameters by *Prop.* 32. and the Square of GE = to the Squares of GD and GF taken together (for the Square of GF *i. e.* GB × □ GD is = □ BD or BA or GE by the *Pythag. Theor.*) and ſo the Circle deſcribed by GE will be = to 2 Circles deſcribed by GD and GF taken together; then taking away the Common Circle deſcribed by GF there will remain the circle deſcribed by GF within the Cone equal to the *Annulus* or Ring deſcribed by DE about the Sphere. And ſince this may be demonſtrated after the ſame way in any other caſe, *viz.* that a circle deſcribed by *g f*, will be equal to an *Annulus* deſcribed by *d e*; it will follow, that all Rings or *Annuli* deſcribed by the Lines DE or *de* (*i. e.* all that Solid that is conceived to be deſcribed by the trilinear Figure ADLM turned round) will be equal to all the Circles deſcribed by GF or *gf* (*i.e.* to the Cone generated by the Triangle BLM;) and ſo as the Cone i $\frac{1}{3}$ part of the Cylinder generated by AL, by the *Conſect.* of *Prop.* 38. ſo alſo the Solid made by the Trilinear ADLM (*viz.* the Exceſs of the Cylinder above the Sphere) will be $\frac{1}{3}$ of the Cylinder, and conſequently the Hemiſphere $\frac{2}{3}$. Q. E. D.

CONSECTAYS.

I. Hence you have a further Confirmation of *Conſect.* 2. *Prop.* 32. and *Prop.* 36. N. 3.

II. Hence alſo naturally flows a Confirmation of *Conſect.* 2. *Definit.* 20. and conſequently the Dimenſion of the Sphere both as to its ſolidity and Surface. For putting a for the Diameter of the Sphere and circumſcribed Cylinder, and Ea for the Circumference, the Baſis of the greateſt Circle will be $\frac{1}{4}eaa$, and that multiplied by the Altitude, gives $\frac{1}{4}ea^3$ for the Cylinder. Therefore by the preſent Propoſition, $\frac{1}{6}ea^3$ gives the Solidity of the Sphere (by making as 3 to 2 so $\frac{1}{4}$ to $\frac{1}{6}$) This divided by $\frac{1}{6}a$, will give, by vertue of *Conſect.* 1. of the aforeſaid *Def.* 20. and *Conſect.* 3. *Definit.* 17. the Surface of the Sphere eaa.

III. Therefore the (α) Surface of the Sphere eaa, is manifeſtly Quadruple of the greateſt Circle $\frac{1}{4}eaa$.

IV. The Surface of the Cylinder, without the Baſes, made by multiplying the Altitude a by the Circular Periphery of the Baſe ea, will be eea, equal to the Surface of the Sphere.

V. Adding therefore the 2 Baſes, each whereof is $\frac{1}{4}eaa$, the whole Surface of the Cylinder $1\frac{1}{2}eaa$, will be to the Surface of the Sphere eaa as 3 to 2.

VI. The Square of the Diameter aa to the Area of the Circle $\frac{1}{4}eaa$, is as a to $\frac{1}{4}ea$, *i. e.* as the Diameter to the 4th part of the circumference.

VII. A Cone of the ſame Baſe and Altitude with the Sphere and Cylinder, will be by *Conſect.* 2. of this, *Prop.* and the *Conſect.* of *Prop.* 38. $\frac{1}{12}ea^3$, and of the Cylinders $\frac{1}{4}$ or $\frac{3}{12}ea^3$. Therefore a Cone, Sphere, and Cylinder, of the ſame heighth and diameter, are as 1, 2, 3. The Cone therefore is equal to the Exceſs of the Cylinder above the Sphere; as is otherwiſe evident in *Scholium* 1. of this.

(α) *Archim.* *lib.*1. *de Sph.* & *Cylind.* *Prop.* 31. (al. 30).

SCHOLIUM II.

AND thus we have briefly and directly demonſtrated the chief Propoſitions of *Archimedes*, in his 1. Book *de Sphær. & Cyl.ind.* which he has deduced by a tedious *Apparatus*, and only indirectly. And now if you have a mind to Survey the larger and more perplext way of *Archimedes*, and compare it with this ſhorter cut we have given you; take it thus: *Archimedes* thought it neceſſary firſt of all to premiſe this *Lemma*; That all the Conical Surfaces of the Conical Body made by Circumvolution of the Polygon, or many-angled Figure A, B, C, D, E, &c. (*Fig.* 82.) inſcribed in a Circle, according to *Definit.* 19. I ſay, thoſe Conical Surfaces taken all together, will be equal to a Circle, whoſe Radius is a mean Proportional between the Diameter AE and a tranſverſe Line BE, drawn from one extremity of the Diameter E to the end of the ſide AB next to the other extremity. This we will thus demonſtrate by the help of ſpecious Arithmetick: Since BN, HN are the Right Sines of equal Arches, CK and CK whole Sines, &c. and the Lines BH, GC, &c. parallel; having drawn obliquely the tranſverſe Lines HC, GD, all the angles at H, C, G, D, &c. will be equal by *Conſect.* 1. *Definit.* 11. and conſequently all the Triangles BNA, HNI, ICK, &c. equiangular, both among themſelves, and to the △ ABE; ſince the angle at B is a Right one, by *Conſect.* 1 *Prop.* 33. and the angle at A common with the △ BNA. Wherefore as BN to NA ſo CK to CI.
or HN to NI ſo & GK to KL.
and ſo DM to ML / FM to ME ſo EB to BA; and ſo by making BN, HN, DM, FM $= a$ CK and GK $= b$, EB $= c$, for NA, NI, ML and ME, you may rightly put $e\,a$ for KI and KL $e\,b$ for *AB*, $e\,c$. Which being done you may eaſily obtain the Conical Surfaces of the inſcrib'd Solid, and the Area of a Circle whoſe Radius ſhall be a mean Proportional between AE and FB, and it will be evidently manifeſt, that theſe two are equal. For, 1. (*for Conical Surfaces*) the Diameter of the *Baſe* $BH = 2\,a$, and the ſide of the Cone $AB = e\,c$: Therefore (making here o the name of the Reaſon between the Diameter and Circumference) the circumference will be $2\,o\,a$

which multiplied by half the ſide $\frac{1}{2}ec$, gives the Conical Surface $oaec$, by *Conſect.* 4. *Definit.* 18. And ſince the Circumference BH is as before $2oa$, and the circumference CG $= 2ob$, half of the ſum $2oa + 2ob$, *viz.* $oa + ob$ is the *equated* Circumference: which multiplied by the Side BC $= ec$ gives the Surface of the truncated Cone BHGC $= oaec - obec$, by *Conſect* 5. of the aforeſaid *Def.* ſince, laſtly, the Surface of the truncated Cone DFGC is equal to one, and likewiſe the conical Surface EDF to the other, by adding you'l have the Sum of all $4oaec + 2obec$.

2. (for the Area of the Circle) the Diameter AE is $= 4ea + 2eb$, and BE $= c$: the Rectangle of theſe is $= 4eac + 2ebc =$ (which alſo is evidently equal to the Rectangle of all the tranſverſe Lines BH, CG, DF into the ſide AB, as *Archimedes* propoſes in the matter) $=$ to the Square of the Radius in the Circle ſought, becauſe the Radius is a mean Proportional between AE and BE, and ſo equal to $\sqrt{4eac + 2ebc}$, ſo that the whole Diameter is $2\sqrt{4eac + 2ebc}$. Therefore, 2. the circumference of this Circle will be $20\sqrt{4eac + 2ebc}$; i. e. $\sqrt{1600eac + 800ebc}$: which multiplyed by half the Semidiameter, *i. e.* by $\frac{1}{2}\sqrt{4eac + 2ebc}$, *i. e.* $\sqrt{eac + \frac{1}{2}ebc}$ gives the Area of the Circle ſought $\sqrt{1600aaeecc + 1600eeabcc + 400eebbcc}$. But this Root extracted is $40aec + 20bec$, equal to the ſuperiour Sum of the conical Surfaces. Q.E.D.

Having thus demonſtrated the Lemma, we will eaſily demonſtrate with *Archimedes* (tho not after his way) That the Surface of any Sphere, is Quadruple of the greateſt Circle in it, which is already evident from the 2d *Conſect.* For ſince all the conical Surfaces of the inſcribed Solid taken together, by the preceding Lemma, are equal to the Area of a Circle whoſe Radius is a mean Proportional between the Diameter AE and the Tranſverſe EB; and this mean Proportional approaches always ſo much nearer to the Diameter AE, and thoſe Surfaces ſo much nearer to the Surface of the Sphere, by how many the more ſides the inſcribed Figure is conceived to have, by *Conſect.* 1 and 2. *Def.* 18. if you conceive in your mind the Biſection of the Arches AB, BC, &c. to be continued *in Infinitum*, it will neceſſarily follow, that all thoſe conical Surfaces will at length end in the Surface of the Sphere it ſelf, and that mean Proportional in the diameter AE, and ſo the Surface of the Sphere will be equal to a Circle, whoſe Radius is the Diameter AE, But that Circle would be Quadruple of the greateſt Circle in the preſent Sphere, by *Prop.* 35. Therefore the Surface of the Sphere is Quadruple of that Circle alſo. Q. E. D.

Hence alſo it would be very eaſie to deduce with *Archimedes* (tho again after another way) that celebrated Propoſition, which we have already demonſtrated from another Principle in the Prop. of this Schol. *viz.* That a Cylinder is to a Sphere of the ſame Diameter and Altitude, as 3 to 2. For by putting a for the Diameter and Altitude, and $e\,a$ for the Circumference, the Area of the Circle, will be $\frac{1}{4}e\,a\,a$; and this Area being multiplied by the Altitude a, gives $\frac{1}{4}e\,a^3$ for the Cylinder, by *Conſect.* 5. *Definit.* 16. and the ſame Quadruple, *i. e.* $e\,a\,a$ multiplied by $\frac{1}{6}a$ gives $\frac{1}{6}e\,a^3$ for the Sphere, by *Conſect.* 1. *Definit.* 20. and *Conſect.* 3. *Definit.* 17. Wherefore the Cylinder will be to the Sphere as $\frac{1}{4}$ to $\frac{1}{6}$, *i. e.* in the ſame Denominator as $\frac{6}{24}$ to $\frac{4}{24}$, *i. e.* as 6 to 4, or 3 to 2. Q.E.D.

Whence it is evident, that the Dimenſion of the Sphere would be every ways abſolute if the Proportion of the Diameter to the Circumference were known; which now with *Archimedes* we will endeavour to Inveſtigate.

Propoſition XL.

THE *Proportion of the Periphery of a Circle* (a) *to the Diameter, is leſs than* $3\frac{1}{7}$ *or* $\frac{10}{70}$ to 1. and greater than $3\frac{1}{71}$ to 1.

Demonſtration.

The whole force of this Propoſition conſiſts in theſe, that, 1. Any Figure circumſcribed about a Circle, has a greater Periphery than the Circle, but any inſcrib'd one a leſs. 2. The Periphery of a circumſcribed Figure of 96 ſides, has a leſs Proportion to the Diameter, than $3\frac{1}{7}$ to 1.

To demonſtrate this ſecond, we will enquire 3. the Proportion of one ſide of ſuch a Figure, whether circumſcribed or inſcribed after the following way.

(a) *Archim. Cyclom. prop.* 2

For the firſt part of the Propoſition.

Suppoſe the Arch *B*C (*Fig.* 84. N. 1.) of 30 degrees, and its Tangent *B*C making with the Radius AC a Right Angle, to make the Triangle ABC half of an Equilateral one, ſo that AB ſhall be to BC in double Reaſon, *viz.* as 1000 to 500; which being ſuppoſed, AC will be the Root of the Difference of the Squares BC and AB, *i. e.* a little greater than 866, but not quite $\frac{3}{100}$.

Then continually Biſecting the Angles BAC by AG, GAC by AH, HAC by AK, KAC by AL, BC is half the ſide of a circumſcribed Hexagon, GC the half ſide of a Dodecagon (or 12 ſided Figure) HC of a Polygon of 24 ſides, KC of one of 48; laſtly, LC of one of 96 ſides; and by N. 3. *Schol.* 3. *Prop.* 34. GC will be to AC as BC to BA + AC, and alſo HC to AC as GC to GA + AC, *&c.* Wherefore

In the firſt Biſection, of what parts GC is 500, of the ſame will AC be 1866 and a little more, and AG (which is the Root of the Sum of the □ □ GC and AC) 1931$\frac{8}{10}$+.

In the ſecond Biſection, of what parts HC is 500, of the ſame will AC be found to be 3797$\frac{8}{10}$+ and AH 3830$\frac{5}{10}$+.

In the third Biſection, of what parts KL is 500 of the ſame will AC be 7628$\frac{3}{10}$+ and AK 7644$\frac{6}{10}$+.

In the 4th Biſection, of what parts LC is 500 of the ſame will AC be 15272$\frac{9}{10}$+.

Now therefore LC taken 96 times, will give 48000 the Semi-periphery of the Polygon, which has the ſame Proportion to the Semi-diameter AC 15272$\frac{9}{10}$ as the whole Periphery to the whole Diameter. But 48000 contains 15272$\frac{9}{10}$, 3 times, and moreover 2181$\frac{3}{10}$ remaining parts, which are leſs than $\frac{1}{7}$ part of the diviſion, for multiplied by 7 they give only 15269$\frac{1}{10}$+.

Therefore it is evident, that the Periphery of this Polygon (and much more the Periphery of a leſs Circle than that) will have a leſs Proportion to the Diameter, than 3$\frac{1}{7}$ to 1. Which is one thing we were to demonſtrate.

For

For the 2d *Part of the* Prop.

Suppoſe the Arch BC to be (N° 2.) of 60 Degr. that is the Angle BAC at the Periphery of 30 Since the Angle at B is a right one by *Conſect.* 1. *Prop.* 33. the Triangle ABC will be again half an Equilateral one, and BC the whole ſide of an Hexagon, and GC of a Dodecagon, &c. So that putting for BC 1000 (as before we put 500 for the ſide of the Hexagon) let AC be 2000 and AB the Root of the difference of the Squares BC and AC *i. e.* leſs than $1732\frac{1}{10}$ *viz.* 1732 and not quite $\frac{6}{100}$.

Then biſecting continually the Angles BAC, GAC, &c. ſince the Angles at the Periphery BAG, GAC, GCB, ſtanding on equal Arches BG and GC, are equal by *Prop.* 33. and the Angle at C, (common to the Triangles GCF and GCA) and the others at H, K, L are all right ones by *Conſect.* 1. of the aforeſaid *Prop.* theſe 2 Triangles CGF and CGA are equiangular and conſequently by *Prop.* 34. the Perpendicular GC in the one will be to the Perpendicular GA in the other as the Hypothenuſe CF in the one to the Hypoth. AC in the other *i. e.* (by the foundation we have laid in the former part of the Demonſtration of N° 3, *Schol.* 3. *Prop.* 34.) as BC to AB+AC; and in like manner in the following HC will be to HA as GC to AG+AC, &c. Wherefore.

In the firſt Biſection, of what parts GC is 1000 of the ſame AG will be a little leſs then $3732\frac{1}{10}$; and AC (which is the Root of the Sum of the ☐☐ AG and GC) will be a little leſs then $3863\frac{8}{10}$.

In the ſecond Biſection, of what parts GC is 1000 of the ſame will AH be a little leſs then $7595\frac{9}{10}$ and AC a little leſs then $7661\frac{5}{10}$.

In the third Biſection, of what parts KC is 1000 of the ſame will AK be a little leſs then $15257\frac{4}{10}$ and AC a little leſs $15290\frac{7}{10}$.

In the fourth Biſection, of what parts LC is 1000 of the ſame will AL be a little leſs then $30547\frac{6}{30}$; and AC a little leſs then 30564, and conſequently if LC be put 500, AC will be leſs then 15282.

Now therefore LC taken 96 times will give 48000 for the Periphery of the inſcrib'd Polygon, and 15282 and a little leſs for the Diameter AC. But 48000 contains 15282 thrice, and moreover a remainder of 2154 parts, which are more then $\frac{10}{71}$ of the Diviſor; for $\frac{1}{71}$ of this number makes $215\frac{17}{71}$ and ſo $\frac{10}{71}$ makes $2150\frac{170}{71}$ *i. e.* $2152\frac{28}{71}$. Therefore it is evident that the Periphery of this Inſcribed Polygon (and much more the Periphery of a Circle greater then that) will be to it's Diameter in a greater proportion then $3\frac{1}{71}$ to 1, which is the 2*d* thing.

SCHOLIUM

IF any one had rather make uſe of the ſmall numbers of *Archimedes*, which he choſe for this purpoſe, by putting in the firſt part of the Demonſtration, for AB 306 and for BC 153, in the ſecond for AC 1560 and for BC 780, by the like proceſs of Demonſtration, he may infer the ſame with *Archimedes*. We like our Numbers beſt, tho' ſomewhat large, becauſe they may be remember'd, and are more proportionate to things, and make alſo the latter part of our Demonſtration like the former. The Proportion in the mean while of the Diameter to the Periphery of the Circle by the *Archimedean* way is included within ſuch narrow Limits, that they only differ from one another $\frac{1}{497}$ or $\frac{10}{4970}$ parts; for $\frac{10}{71}$ Subtracted from $\frac{10}{70}$ leave $\frac{1}{497}$, as $3\frac{10}{71}$ or $\frac{223}{71}$ and $3\frac{16}{70}$ or $\frac{220}{70}$ if they are reduced to the ſame Denomination make on the one hand $\frac{15610}{4970}$ on the other $\frac{15620}{4970}$. Hence it would be eaſy, having divided the difference $\frac{10}{4976}$ into 2 Parts, to expreſs a middle proportion of the Periphery to the Diameter between the 2 Archimedean and Extreme ones as in theſe Numbers, 1561 to 4970, (or by dividing both ſides by 5) as 3123 to 994, or (dividing both ſides by 7) as $446\frac{1}{7}$ to 142, or (by dividing again by 2) as $223\frac{1}{14}$ to 71, &c.

While theſe Numbers become as fit for uſe as thoſe of *Archimedes*, which we therefore uſe before any other, particularly in Dimenſions that dont require an exact Niceneſs; where they do thoſe may be made uſe of exhibited by *Ptolomey Vieta*, *Ludolphus a Ceulea*, *Metius*, *Snellius*, *Lansbergius*, *Hugeus*, &c. as if

The Diameter be	The Circumference will be
10, 000, 000	31, 416, 666. *Ptolomey.*
10 000, 000, 000	31, 415, 926, 535. *Vieta.*
100, 000, 000, 000, 000, 000 000	314, 159, 265, 358, 979, 323, $846\frac{1}{2}$, &c. *Ludolph. a Ceulen.*

Propoſition XLI.

THe Area of a Circle has the ſame proportion to the Square of it's Diameter, as the 4th *part of the Circumference to the Diameter.*

Demonſtration.

Though we have before Demonſtrated this Truth in *Conſect.* 6. *Prop.* 39. yet here we will give it you again after another way. Since therefore the Circumference is a little leſs then $3\frac{1}{7}$, and a little more than $3\frac{10}{71}$ Diameters, for this exceſs putting z if the Diameter be 1 we will call the Circumference $3+z$ therefore the 4*th* Part of it will be $\frac{3+z}{4}$: And the Area of the Circle (by Multiplying the half Semidiameter) *i. e.* $\frac{1}{4}$ by the Circumference, you'l have both $\frac{3+z}{4}$ and the ſquare of the Diameter $= 1$. Q. E. D.

CONSECTARY.

THerefore if the (α) Proportion of *Archimedes* be near enough truth to be made uſe of, *viz.* 22 to 7; the Area of the Circle will be to the *S*quare of the Diameter as 11 to 14, becauſe the quarter part of 22, *i. e.* $5\frac{1}{2}$ or $\frac{11}{2}$ to the Diam. 7 $\frac{14}{2}$ is in the ſame Proportion.

Propoſition XLII.

THE Diameter (β) of a Square AC (*Fig.* 83.) is incommenſurable to the ſide AB (and conſequently alſo to the whole Periphery) *i. e.* it bears a Proportion to it that cannot be exactly expreſſed by Numbers.

(α) *Archim. Prop.* 3. *Cyclom.*

(β) *Eucl. laſt Prop. lib.* 10.

Demonſtration.

For, if for AB you put 1, BC will be alſo

1,

1, and by the *Pythagorick Theorem* AC will be $=\sqrt{2}$. Therefore by *Conſect.* 4. *Definit.* 30. AC is incommenſurable to the ſide AB, *&c.* Q.E.D.

CONSECTARY I.

IT is notwithſtanding Commenſurable in Power; for its Square is to the Sqnare of the Side as 2 to 1.

CONSECTARY II.

NOW if the Proportion of the ſide or whole Periphery ABCDA, to the Diam. AC is to be expreſſed by Numbers ſomewhat near, as we have done in the Diameter and Circumference of a Circle; then making the ſide $AB=100$, the Diameter is greater than $141,\frac{42}{100}$, leſs than $141,\frac{43}{100}$.

Propoſition XLIII.

THE Area of a Circle is Incommenſurable to the Square of the Diameter.

Demonſtration.

For dividing the Semidiameter CD (*Fig.* 85.) into two equal Parts (and conſequently the Diameter DF into 4) AC will be 2, *viz.* $\sqrt{4}$ and $\mathfrak{AC}\ \sqrt{3}$. by *Schol.* 2. *Prop.* 34. N. 3. the ſum $\sqrt{4}+\sqrt{3}$, and the ſum of as many equal to the greateſt AC 4. Having moreover Biſected the Parts of the Semidiameter, AC will $=4$ or the $\sqrt{16}$, $ac=\sqrt{15}$, $\mathfrak{AC}=\sqrt{12}$, $\mathfrak{AC}=\sqrt{7}$; the ſum $\sqrt{16}+\sqrt{15}+\sqrt{12}+\sqrt{7}$; and the ſum of as many equal to the greateſt AC is $=16$, *&c.* And thus the laſt ſums will be the Square Numbers increaſing in Quadruple Proportion; but the former Sums will be always compoſed of the Rational Root of every ſuch Square, and of ſeveral other irrational Roots of Numbers unevenly decreaſing; ſo that it will be impoſſible to expreſs thoſe former Sums by any Rational Number, by what we have ſaid in *Schol.* 2. *Definit.* 30. Wherefore all the Indiviſibles of the Quadrant ADC are to as many of the Square ACDE equal to the greateſt, *i. e.* the Quadrant it ſelf

ADC to the Square ACDE (and conſequently the whole Area of the Circle to this circumſcrib'd Square) will be as a Surd Quantity to a true and truly Square Number, *i. e.* the Area of the Circle will be Incommenſurable to the Square of the Diameter, by *Conſect.* 4. of the ſaid *Definit.* Q. E. D.

CONSECTARY.

AND becauſe the fourth part of the Circumference has the ſame Proportion to the Diameter, as the Area of the Circle to the Square of the Diameter, by Prop. 41. Therefore alſo that will be Incommenſurable to this, and conſequently the whole Circumference will be ſo to the Diameter.

SCHOLIUM.

WHerefore it is ſomewhat Wonderful, which G. G. *Leibnitius* (α) tells us, that the Square of the Diameter being 1, the Area of the Circle will be $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$, &c. *ad Infinitum. i. e.* by adding $1\frac{1}{5}-\frac{1}{3}$ and $+\frac{1}{9}-\frac{1}{7}$, &c.) to $\frac{2}{3}+\frac{2}{35}+\frac{2}{99}$, &c. *i. e.* to the Sum of infinite Fractions whoſe common Numerator is 2. But their Denominators Squares leſſen'd by Unity, and taken out of the Series of the Squares of Natural Numbers by every fourth, omitting the Intermediate ones: Which Sum might ſeem expreſſible in Numbers, ſince all its parts are Fractions reducible to a common Denominaton; while notwithſtanding *Leibnitius* himſelf confeſſes, that the Circle is not Commenſurable to the Square, nor expreſſible by any Number.

(α) *Acta Erudit. Ann.* 82. *p.* 44 & *the following.*

CHAP. VII.

CHAP. VII.

Of the Powers of the Sides of Triangles, and other Regular Figures, &c.

Propoſition XLIV.

In Right-angled Triangles (α) (*ABG*, Fig. 86.) *the Square of the Side* (*BC*) *that ſubtends the Right-angle, is equal to the Squares of the other Sides* (*AB and AC*) *taken together.*

Demonſtration.

Though we have demonſtrated this Truth more than once in the foregoing Propoſition; yet here we will confirm it again as follows: Having deſcribed on each ſide of the Square BE a Semicircle, which will all neceſſarily touch one another in one point, and be equal to the Semicircle, BAC, if you conceive as many Triangles inſcribed alſo equal to BAC; it will be evident that the Square BE will contain the ſaid 4 Triangles; and beſides the little Square FGHI, whoſe ſide FI, *v. g.* is the difference between the greater ſide of the Triangle CI, and the leſs CF, (for becauſe the leſs ſide CF=BA, lying in the firſt Semicircle, if it be continued to I in the ſecond Semicircle makes CI=CA the greater ſide of the other Triangle, and ſo in the others. From thence it is evident, That as the Angles ABC, and ACB together make one right one; ſo likewiſe BCF (=CBA) and ECF make alſo one right one; and conſequently ECF is= ACB, and the Arch and the Line EI=to the Arch and the Line AB, &c.) Wherefore, if the greateſt ſide of the given Triangle BC or BD, &c. be called a, and AC, b and the leaſt AC, or CF, &c. be called c; the of the ſide BC, will be=aa, and the Area of each Triangle $\frac{1}{2}\,b\,c$: and ſo the 4 Triangles together $2\,bc$: but the ſide of the middle little Square will be $b—c$, and its Square $bb + cc = 2\,bc$: Wherefore if you add to this the 4 Trian-

(α) *Euc.* 47. *Lib.* 1.

The Sum of all, *i.e.* the whole Square BE will be $bb + cc = aa$. Q. E. D.

CONSECTARYS.

I. HEnce having the ſides that comprehend the Right-angle given, $AC = b$ and $AB = c$, the Hypothenuſe or Baſe that ſubtends the Right-angle BC will be $= \sqrt{bb + cc}$.

II. But if BC be given $= a$ and $AC = b$, and you are to find $AB = x$; becauſe $xx + bb = aa$; you'l have (taking away from both ſides bb) $xx = aa - bb$: therefore x, *i. e.* $AB = \sqrt{aa - bb}$.

III. If 2 Right-angled Triangles have their Hypothenuſa's and one Leg equal, the other will alſo be equal.

Propoſition XLV.

IN Obtuſe-angled Triangles (Fig. 87. N. 1. the Square of the Baſ or greateſt Side BC that ſubtends the Obtuſe-angle BAC, is equa to the Squares of (α) the other 2 Sides (AB and AC) taken together and alſo to 2 Rectangles (CAD) made by one of th Sides which contain the Obtuſe-angle (AC) and it continuation AD to the Perpendicular BD let fall fro the other ſide.

(α) Eucl. prop. 12. lib. 2.

Demonſtration.

If BC be called a, $AB = c$, $AC = b$, $AD = x$, CD will b $= b + x$. Therefore □ $BD = cc - xx$, by *Conſect.* 2. of the preceding *Prop.* In like manner if □ $CD = bb + 2bx + xx$ be ſubtracted from the □ $BC = aa$, you'l have $aa - bb - 2bx - xx =$ to the ſame □ BD. Therefore

$cc - xx = aa - bb - 2bx - xx$,

i. e. (adding on both ſides xx)

$cc + aa - bb - 2bx$.

i. e. (adding on both ſides bb and $2bx$)

$cc + bb + 2bx = aa$. Q. E. D.

CONSECTARY.

IF in this laſt Equation you ſubtract from both Sides $cc+bb$ then will $2bx=aa-bb-cc$ and (if you moreover divide both Sides by $2b$) you'l have $\frac{x=aa-bb-cc:}{2b}$ Which is the Rule, when you have the Sides of an Obtuſe-angled Triangle given, to find the Segment AD, and conſequently the Perpendicular BD.

Propoſition XLVI.

IN *Acute-angled Triangles* (α) *the Square of any ſide* (e.g. *B.C.* Fig. 87. N. 2.) *ſubtending any of the Angles, as A is equal to the Squares of the other* 2 *ſides* (*AB and AC*) *taken together, leſs* 2 *Rectangles* (*CAD*) *made by one ſide, containing the Acute-angle* (*CA*) *and its Segment AD reaching from the Acute-angle* (*A*) *to the Perpendicular* (*BE*) *let fall from the other ſide.*

Demonſtration.

Make again BC$=a$, AC$=b$, AB$=c$, AD$=x$; then will CD$=b-x$. Therefore $cc-xx=\square$ BD, and $aa-bb+2bbx-xx$ (*i. e.* $\square$ BC$-\square$CD) will alſo be $=\square$BD.

Therefore $cc-xx=aa-bb+2bx-xx$.

i. e. (adding to both ſides xx)

$cc=aa-bb+2bx$,

i. e. (adding on both ſides bb, and ſubtracting $2bx$)

$cc+bb+2bx=aa$. Q. E. D.

CONSECTARYS.

I. IF in the laſt Equation, except one, you add on both ſides bb, and ſubtract aa, you'l have $cc+bb-aa==2bx$, and, if moreover you divide both ſides by $2b$, you'l have

$$\frac{cc+bb-aa}{2b}=x:$$

Which is the Rule, having 3 ſides given in an Acute-angled Triangle, to find the Segment AD, and conſequently the Perpendicular BD.

(α) *Eucl. Prop.* 13. *l.* 2.

Knowing therefore the Segments AD and CD, and alſo th Perpendicular BD in Oblique-angled Triangles, whether O tuſe-angled or Acute-angled, when moreover the ſides BC an AB are likewiſe given, the Angles of either Right-angled Tr angles or Oblique-angled ones, will be known; ſo that the la Caſe of Plane Trigonometry, which we deferr'd from *Prop.* 3 to this place, may hence receive its ſolution.

Propoſition XLVII.

THE Square of the Tangent of a (α) *Circle, is equal to a Rectang contain'd under the whole Secant DA, and that part of it whic is without the Circle DE, whether the Secant paſs thro' the Centre not.*

Demonſtration.

(α) *Eucl.* 3 *Lib.* 3.

For in the firſt Caſe, if CB and CE are $= b$, DE $= x$, then will CD $= b + x$, & AD $= 2b + x$: therefore

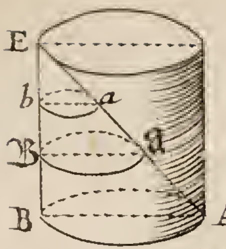
□ ADE $= 2bx + xx$ & □ CD $= bb + 2bx + xx$. ther fore, if from the □ CD you ſubſtract □ CB $= bb$ the remainde will be $2bx + xx =$ □ BD $=$ □ ADE. Q. E. D.

In the ſecond Caſe, the lines remaining as before, make D $= y$, FE or FA $= Z$: therefore the □ ADE will be $= 2z$ $+ yy$, but the □ FC equal to the □ EC $-$ □ FE $= bb - z$ *i. e.* $2bx + xx + zz$. But the ſame Square FD is $= z$ $+ 2zy + yy$. Wherefore taking away from theſe equal Squar the common one zz you'l have $2zy + yy = 2bx + x$ *i. e.* pr. 1ſt Caſe $=$ □ BD. Q. E. D.

CONSECTARYS.

I. THerefore the Rectangles of diverſe ſecants (as of AD & *a de* in *Fig.* 88, *n.* 1.) which are equal to the ſam Square BD, are equal alſo to one another; which the la Equation in our Demonſtration ($2zy + yy = 2bx + xx$ is an ocular proof of.

II. There



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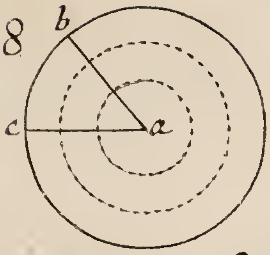

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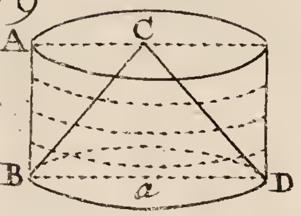

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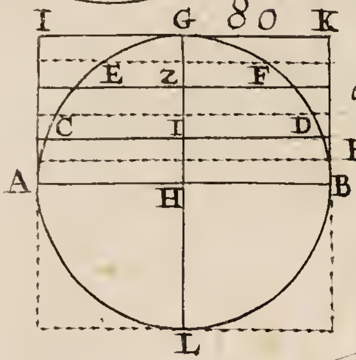

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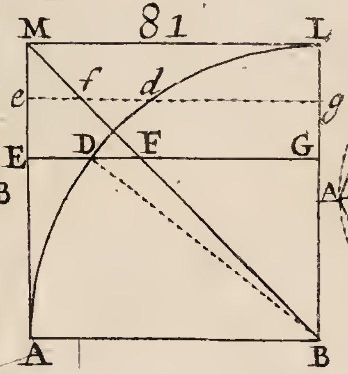

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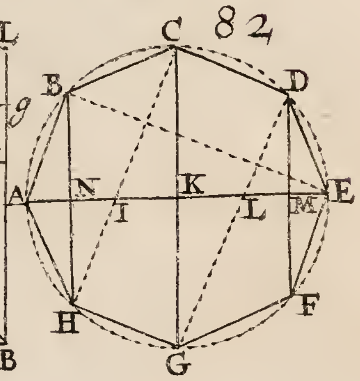

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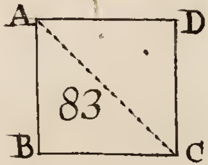

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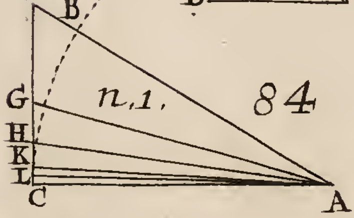

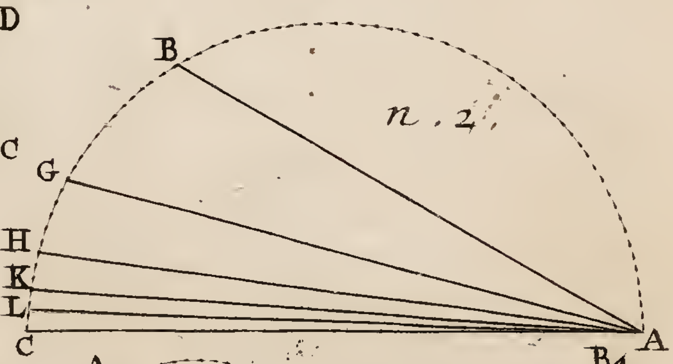

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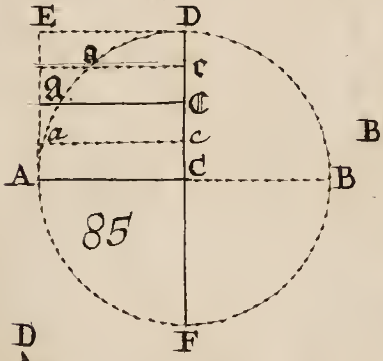

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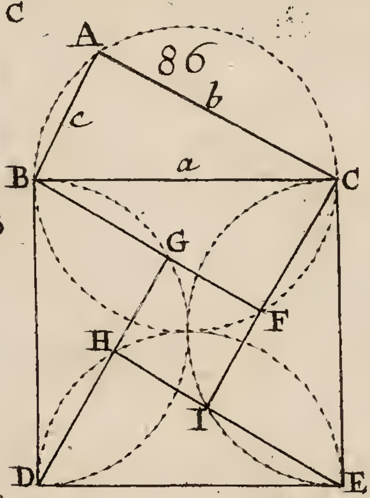

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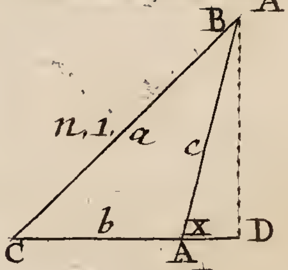

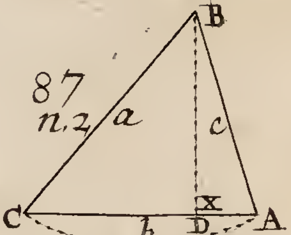

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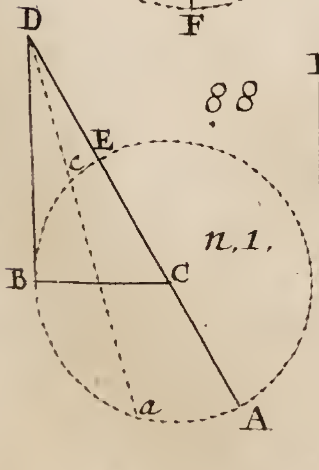
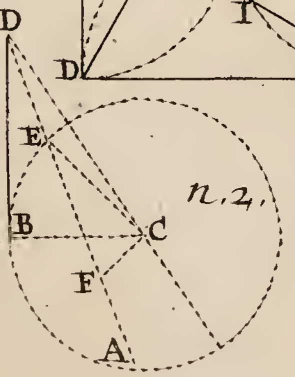
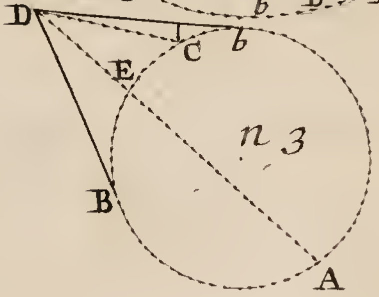






II. Therefore by *Prop.* 19. as aD to AD ſo is reciprocally DE to D *e* in the Fig. of the 2 Caſe.

III. Tangents to the ſame Circle from the ſame Point, as DB & D *b* (*n.* 3.) are equal; becauſe the Square of each is equal to the ſame Rectangle.

IV. Nor can there be more Tangents drawn from the ſame point then two: For if beſides DB & D *b*, D *c* could alſo touch the Circle, then it would be equal to them. By *Conſ.* 2. but that is abſurd by *Conſ.* 3. *Def.* 7.

SCHOLIUM I.

HEnce is evident, the Original of the Geometrical Conſtructions, which *Cartes* makes uſe of, p. 6, and 7. in reſolving theſe 3 Equations, $ZZ=aZ+bb$, $ZZ=aZ+bb$, & $ZZ=aZ=bb$. For in the Firſt caſe, ſince he makes N O or N L (*Fig.* 89. 11. 1.) or NP $\frac{1}{2}a$, and the Tangent $Z\,M\,b$, make MNO through the Center *xx* will $=Z$ the quantity ſought; which thus appears: Making $MO=Z$, NM will $=Z\ \frac{1}{2}a$, and its $\square\ ZZ-aZ+\frac{1}{4}aa$. But the $\square$ OMP (which is by the preſent Prop. $=\square LM$, *i. e.* bb) together with the $\square$ NL or NP *i. e.* $\frac{1}{4}aa$ is $=\square$ NM by the *Pathagor. Theor.* Therefore $ZZ-aZ+\frac{1}{4}aa=bb+\frac{1}{4}aa$. *i e.* (taking away from both ſides $\frac{1}{4}aa$) $ZZ-aZ=bb$ *i. e.* (by adding on both ſides aZ) $ZZ=aZ+bb$: Which is the Equation propoſed. In the Second Caſe, if you make $PM=Z$ (as *Cartes* makes it) you'll have $\square NM+ZZ+aZ+\frac{1}{4}aa$, and to this again as before you'l have $=bb+\frac{1}{4}aa$. Therefore $ZZ+aZ=bb$: Therefore $ZZ=-aZ+bb$: Which is the very Equation of the ſecond Caſe. In the third Caſe, whether you make the whole Secant RM (*N.* 2) or that part of it without the Circle $QM=Z$, the Root ſought, there will come out on both ſides the ſame Equation of the third Caſe; and ſo it is manifeſt, that this Equation has thoſe 2 Roots. For if RM be $=Z$ (adding + to the Fig of Cartes the Line NO which ſhall biſect QR, and makes $OM=LN$) OR or OQ will be $=Z-\frac{1}{2}a$, and ſo the $\square\ OQ=ZZ-aZ+\frac{1}{4}aa$, and this together with the $\square$ RMQ (which is by vertue of the preſ. Prop. $=\square LM$) $=\square NQ$. *i. e.* OM, *i. e.* $ZZ-$

$aZ + \frac{1}{4}aa + bb = \frac{1}{4}aa$, *i. e.* (by adding aZ and taking away $\frac{1}{2}aa$) $ZZ + bb = aZ$; *i. e.* (taking away bb) $ZZ = aZ - bb$: Which is the the very Equation of the third Caſe. But if QM be made $= Z$, OQ or OR will $= \frac{1}{2} - Z$, and its $\square = \frac{1}{4}aa - aZ + ZZ$, as well as the former, and ſo all the reſt. Q. E. D.

SCHOLIUM. II.

NOW if you would immediately deduce theſe Rules by the preſent Propoſition, without the *Pythagorick Theorem.* It may (*e. g*, in the firſt Caſe) be done much ſhorter thus: If $MO = Z$ and NO or $NP = \frac{1}{2}a$, then will $PM = Za$: Therefore ▭ $OMP = ZZ - aZ = bb$, or the $\square$ LM, by the preſ. Propoſition; by adding therefore to both ſides aZ, you'l have $ZZ = aZ + bb$, which is the very Equation of the firſt Caſe. In the ſecond Caſe, if MR be Z, QM or PR will $= a - Z$: Therefore ▭ RMP $aZ - ZZ = bb$, *i. e.* $aZ = bb + ZZ$, *i. e.* $aZ - bb = ZZ$; but if $QM = Z$, RM will $= a - Z$. Therefore ▭ RMP $aZ - ZZ = bb$, as before, *&c.*

SCHOLIUM III:

FRom the 2 Conſect. of the preſent Propoſition, flows another Rule for ſolving the laſt Caſe of *Plain Trigonometry*, which we ſolved in the Conſect. of the foregoing Prop. *viz.* If you have all the three ſides of the Oblique-angled Triangle BCD (*Fig.* 90) given, if from the Center C, at the diſtance of the leſſer ſide CB you deſcribe a Circle, then will, by Conſect. 2. of the preſent Propoſition, BD the Baſe of the Triangle (here we call the greateſt ſide of the Triangle the Baſe, or in an Equicrural Triangle, one of the greateſt) will be to AD, (the ſum of the Sides DC+CB) as DE the differenceof the Sides, to DF the Segment of the Baſe without the Ciecle; which being found, if the remainder of the Baſe within the Circle be divided into two equal parts, you'l have both FG and GB, as alſo DG; which being given, by help of the Right-angled △△ GBC and GDC all the Angles required may be found.

Propoſition XLVIII.

IN any Quadrilateral Figure (α) *ABCD* (Fig. 91. N. 1.) *inſcribed in a Circle, the* ▭ *of the Diagonals AC and BD is equal to the two Rectangles of the oppoſite ſides AB into CD, and AD into BC.*

Demonſtration.

Having drawn AE ſo that the Angle BAE ſhall be equal to the Angle CAD, the Triangles thereby formed (for they have the other Angles EBA and ACD in the ſame Segment equal, by vertue of Conſect. 1. Prop. 33.) will be Equiangular one to another and conſequently (by Prop. 34) as AC to AD ſo *AB* to *B*E. Wherefore by making $AC = a$ and $CD = ea$, and $AB = b$, BE will be $= eb$. In like manner when in the △ △ *B*AC and EAD, the reſpective Angles are equal (*viz.* adding the common part EAF to *B*AE and CAD, equal by Conſtr.) and beſides the angles *B*CA and EDA in the ſame Segment are alſo equal; theſe Triangles will alſo be Equiangular, and AD will be to DE as AC to C*B*; wherefore by putting, as before, a for AC, and oa for C*B*, and c for AD, DE will $= oc$. Therefore the whole $BD = eb + oc$. The Rectangle therefore of AC into *B*D will $= eba + oca =$; the Rectangle of AB into CD $= eba +$ ▭ of AD into $BC = oac$. Q. E. D.

(α) *Ptol.lib.*1. *Almageſh.*

SCHOLIUM

IN Squares and Rectangles (*N.* 2.) the thing is ſelf-evident; For in Squares if the ſide be a, the Diagonals AC and *B*D will be $\sqrt{2aa}$, and ſo their Rectangle $= 2aa$ will be manifeſtly equal to the two Rectangles of the oppoſite ſides. In Oblongs, if the two oppoſite ſides are a and the others b, the Diagonals will be $\sqrt{aa+bb}$, and their Rectangle $aa + bb$ manifeſtly equal to the two Rectangles of the oppoſite Sides.

Propoſition XLIX.

THe ſide (*AB*) *of an Equilateral Triangle* (*ABC*, Fig. 92. *N.* 1.) *inſcribed in a* (α) *Circle, is in Power triple of the Radius* (*AD*) i. e. of the □ *of AD.*

Demonſtration.

M*Ake AD* or *FD*$=a$, and ſo its Square aa. Since Therefore, having drawn DF thro' the middle of A*B*, or the middle of the Arch AFB. let DE be $=\frac{1}{2}a$; for the angles at E are right ones, by *Conſect.* 5. *Definit.* 8. and the Hypothenuſes AD, AF, are equal, by Schol. of Definition 15. but the ſide AE is common. Therrefore the other Sides FE and E*D* are equal by, by *Conſect.* 3. *Prop.* 43.) and the □ of the latter is $\frac{1}{4}aa$, which ſubtracted from aa leaves $\frac{3}{4}aa$ for the □ of AE. Therefore the line AE is $\sqrt{\frac{3}{4}aa}$, and conſequently AB $2\sqrt{\frac{5}{4}aa}$, *i. e.* $\sqrt{\frac{12}{4}aa}$, *i. e.* $\sqrt{3aa}$: therefore □ A*B* $=3aa$. Q. E. D.

CONSECTARYS.

I. IF the Radius of a Circle be $=a$, the ſide of an Inſcribed Regular Triangle will be $\sqrt{3aa}$, *e. g.* if AD be 10, A*B* will be $\sqrt{300}$; and if AD be 10, 000, 000, A*B* will be $\sqrt{300,000,000,000,000}$, *i. e.* 17320508, and the Perpendicular DE 5000, 000.

II. Hence it is evident, that in the geneſis of a *Tetraedrum* propoſed in *Def.* 22, that the elevation CE (*Fig.* 44, *N.* 1.) is to the remaining part of the Diameter of the Sphere CF as 2 to 1; for making the Radius C*B*$=a$ and its □ aa, the □ of A*B* or *B*E will $=3aa$, by the preſent Propoſition. Therefore the □ of C*B* being ſubtracted from the □ of *B*D or *B*E, there remains the □ of CE $=2aa$. *But* ſince CE, C*B*, CF, are continual Proportionals, by *N.* 3. *Schol.* 2. *Prop.* 34. CE will be to CF as the □ of CE to the *S*quare of C*B*, by vertue of Prop. 35. *i. e.* as 2 to 1.

(α) *Eucl.* 12 *lib.* 12.

SCHOLIUM.

HEnce you have the *Euclidean* way of generating (α) a *Tetraedrum*, and inſcribing it in a given Sphere, when he bids you divide the Diameter EF of a given Sphere ſo that EC ſhall be 2 and CF 1, and then at EF to erect the Perpendicular CA, and by means thereof to deſcribe the Circle A*B*D, and to inſcribe therein an Equilateral Triangle, &c.

Propoſition

Propoſition L.

THE *Side* (*AB*) *of a Regular Tetragon* (*or Square*) (*ABCD*, Fig. 92. N. 2.) *is double in Power of the Radius* (*AD*).

Demonſtration.

For having drawn the Diameter AC and *B*D, the Triangle AO*B* is Right-angled, and conſequently, by the *Pythagorick Theorem*, if the □ of AO and *B*O be made equal to aa, then will the □ of $AB=2aa$, Q. E. D.

CONSECTARY.

THerefore when the Radius of the Circle AO is made $= a$, the ſide of the □ A*B* will $= \sqrt{2aa}$, *e.g.* if AO be 10, A*B* will be $\sqrt{200}$; and if AO be 10, 000, 000, A*B* will be $\sqrt{200, 000, 000, 000, 000}$, *i. e.* 14142136.

Propoſition LI.

THE *ſide AB of a Regular Pentagon* (α) (*ABCDE*) (Fig. 93. N. 1.) *is equal in Power to the ſide of an Hexagon and Decagon inſcribed in the ſame Circle,* i. e. *the* □ *of AB is equal to the Squares AF and AO taken together.*

Demonſtration.

Make $AO = a$ and $AF = b$, $AB = x$: We are to demonſtrate that $xx = aa + bb$: which may be done by finding the ſide A*B* by the parts *B*H and HA, after the following way: Firſt of all the angle AO*B* is 72°, and the others in that Triangle at A and *B* 54°. But *B*GG is alſo 54°, as ſubtending the Decagonal Arch *BF* of 36°, and alſo one half of it FG of 18°. Therefore the △△ A*B*O and H*B*O are Equiangular, and you'l have

(α) *Eucl. Prop.* 13. *lib.* 13.

(α) *Eucl. Prop.* 10. *lib.* 13.

As *AB* to *BO* ſo *BO* to *BH*

$$x - a - a - \frac{aa}{x}$$

Secondly, in the Triangle *BFA* the Angles at *B* and A are equal by *N.* 3. *Conſect.* 5. *Def.* 8. and by vertue of the ſame alſo the Angles at F and A in the △FHA are ſo too. Wherefore the △△ *BFA* and FFA are Equiangular, and you'l have

As *BA* to AF ſo AF to AH

$$x \text{ to } b - b - \frac{bb}{x}$$

Therefore the whole line A*B* (becauſe the part AH is found $= \frac{bb}{x}$ and *B*H $= \frac{aa}{x}$) will be $\frac{aa + bb}{x}$, which was firſt made $= x$; ſo that now $\frac{aa + bb}{x}$ is $= x$, and multiplying both ſides by x, $aa + bb = xx$. Q.E.D.

CONSECTARY I.

Therefore if the Radius of a Circle be (a) the ſide of a Pentagon *AB* will be $\sqrt{aa \times bb}$.

CONSECTARY II.

Therefore the □ AI $= \frac{aa + bb}{4}$ and □ OI $=$ □OA $-$ □AI $= aa - \frac{aa - bb}{4}$ i.e. $\frac{3aa - bb}{4}$. Therefore OI $= \sqrt{3aa - bb}$: Which yet may be expreſſed otherwiſe, *viz.* OI $= \frac{a + b.}{2}$

Demonſtration.

(α) *Eucl. Prop.* I. *lib.* 14.

Make (α) OA or OF (*N.* 2.) as before $= a$, AF $= b$, and FI now $= x$; then will OI $= a - x$ and having drawn the Arch FK at the Interval AF, ſo that AK may be equal to this, and FI $=$

IK$=x$; then will the angle IKA$=$F72°. Therefore the angle AKO$=108$; and ſince KOA is 36°, KAO will be alſo 36°, and ſo KA$=$KO$=$AF$=b$. Wherefore OI is $=b+x$, which was above $a-x$. Therefore 2OI$=a-x+b+x$, i. e. $a+b$. Therefore OI$=\frac{a+b}{2}$.

CONSECTARY III.

THerefore the difference between the Perpendicular of the Triangle DE (*Fig.* 92 and 93. *N.* 1.) and the Perpendicular of the Pentagon OI is $=\frac{1}{2}b$, by vertue of *Conſect.* 2. of this and of the Demonſtrat. of *Prop.* 49.

CONSECTARY IV.

HEnce is alſo evident, by the preſent Propoſition, that which in *Fabri*'s Geneſis of an *Icoſaedr. Def.* 22. we ſaid, *viz.* that (See *Fig.* 46.) B*a* is equal to the ſide of a Pentagon *B*A, becauſe, *viz.* *Fa* is $=$ to the Semidiameter OB, and BF is the ſide of a Decagon.

Propoſition LII.

T*HE ſide of an Hexagon is in Power equal to the Radius, as being it ſelf equal to it by* N. I. Schol. Def. 15.

Propoſition LIII.

T*HE ſide of a Regular Octagon (ABCD, &c. Fig. 94.) is equal in Power to half the ſide of the Square, and the difference (PB) of that half ſide from the Radius, taken together.*

Demonſtration.

For that the □ of AB is $=$ to the □ of AP $+$ □ BP, is evident from the *Pythag. Theor.* But that PO is $=$PA half the ſide of the Square, is evident from the equality of the Angles PAO and POA, ſince each is a half right one or 45°. Wherefore the ſide of the Octagon is equal in Power to half the ſide of the Square, *&c.* Q.E.D.

CONSECTARY.

Therefore, if the Radius be $= a$, AP will be, by vertue of the *Pythag. Theor.* $= \sqrt{\tfrac{1}{2} a a}$ and $PB = a - \sqrt{\tfrac{1}{2} a a}$ of that $\square \tfrac{1}{2} a a$; and of this $\square\ 1 \tfrac{1}{2} a a - \sqrt{2}$. Therefore the Sum of $\square$ $AB = 2 a a - \sqrt{2 a^4}$. Therefore the ſide of the Octagon $= \sqrt{2 a a - \sqrt{2 a^4}}$. *e. g.* If AO be 10, AB will be $\sqrt{200 - \sqrt{20000}}$, and if the Radius AO 10000000, AB will be $\sqrt{200,000,000,000,000 - \sqrt{2000000000000000000,00\,00000000000}}$, by vertue of the preſent Conſ. or from the Prop. it ſelf, if AO be 10000000, AP will be vertue of the Conſ. of Prop. 50. $= 7071068$, and conſequently $BP = 2928932$ The Squares of theſe added together give the $\square$ AB, and the Root thence extracted $AB = 7653\,668$.

Propoſition LIV.

The ſide of a Regular Decagon (α) is equal in Power to the greateſt part of the ſide of an Hexagon cut in mean and extreme Reaſon.

Demonſtration.

(α) *Eucl.* 9. *lib.* 13. *Corol.*

Suppoſe BD (*Fig.* 95.) divided in mean and extreme Reaſon in E, and BA to be joined to it long ways $=$ to the ſide of a Decagon inſcribed in the ſame Circle, whoſe Radius is BC or $AC = BD$. Now we are to demonſtrate that DE the greateſt part of the ſide of the Hexagon *B*D divided in mean and extreme Reaſon, is equal to the ſide of a Decagon *B*A, and the Power of the one equal to the Power of the other. Becauſe the Angle AC*B* is 36°, A*B*C and A 72°, and conſequently C*B*D 108°, *B*CD and D will be each 36°, and ſo the whole ACD 72°. (that ſo CD may paſs preciſely thro' the other end of the ſide of the Pentagon AF.) Wherefore the $\triangle\triangle$ ABC and ADC are Equiangular, and

$$AD \text{ to } \left.\begin{matrix}AC\\ \textit{i. e. } BD\end{matrix}\right\} \text{ as } \left.\begin{matrix}AC\\ \textit{i.e. } BD\end{matrix}\right\} \text{ to } AB.$$

Therefore the whole line AD is divided in mean and extreme Reaſon. But *B*D is alſo divided in the ſame Reaſon by Hyp. Wherefore

As AD to D*B* and D*B* to BA,
So DB to DE and DE to E*B*.

Therefore D*B* is in the ſame Proportion to *D*E as D*B* to *B*A. Therefore DE is $=$ *B*A, and the Power of the one to the Power of the other. Q. E. D.

CONSECTARYS.

THerefore, if the Radius of the ſide of the Hexagon is a the ſide of the Decagon will be $\sqrt{\frac{5}{4}aa} - \frac{1}{2}a$, by *Schol.* 2. *Prop.* 27. *e. g.* if the Radius be 10, the ſide of the Decagon will be $\sqrt{125} - 5$, and if the Radius be put 10 000 000, the ſide of the Decagon will be $= \sqrt{125\ 000\ 000\ 000\ 000} - 5000\ 000$, *viz.* by adding the Square of the Radius and the Square of half the Radius into one Sum; whence you'l have the ſide of the Decagon $= 6180340$; the half whereof 3090170 gives the difference between the Perpendiculars of the Triangle and the Pentagon, by *Conſ.* 3. *Prop.* 51.

II. The ſide therefore of the Pentagon is by *Prop.* 51. $\sqrt{\frac{5}{2}aa - \sqrt{\frac{5}{4}a^4}}$; for the Square of the Hexagon is aa or $\frac{2}{2}aa$, the □ of the Decagon $\frac{3}{2}aa - \sqrt{\frac{5}{4}a^4}$: the Root extracted out of the Sum of theſe is the ſide of the Pentagon, *viz.* $\sqrt{\frac{5}{2}aa - \sqrt{\frac{5}{4}a^4}}$ *e. g.* if the Radius be 10, the ſide of the Pentagon will be $\sqrt{250 - \sqrt{125000}}$, and if the Radius be put 10 000 000, ſince the ſide of the Hexagon is equal to it, and the ſide of the Decagon 6180340, their Squares being added into one Sum, the Root extracted out of that Sum will give the ſide of the Pentagon, 1755704 nearly; and the ſides being collected into one ſum, the half of it 8090170 will give the Perpendicular in the Pentagon OI, by *Conſect.* 2. *Prop.* 51.

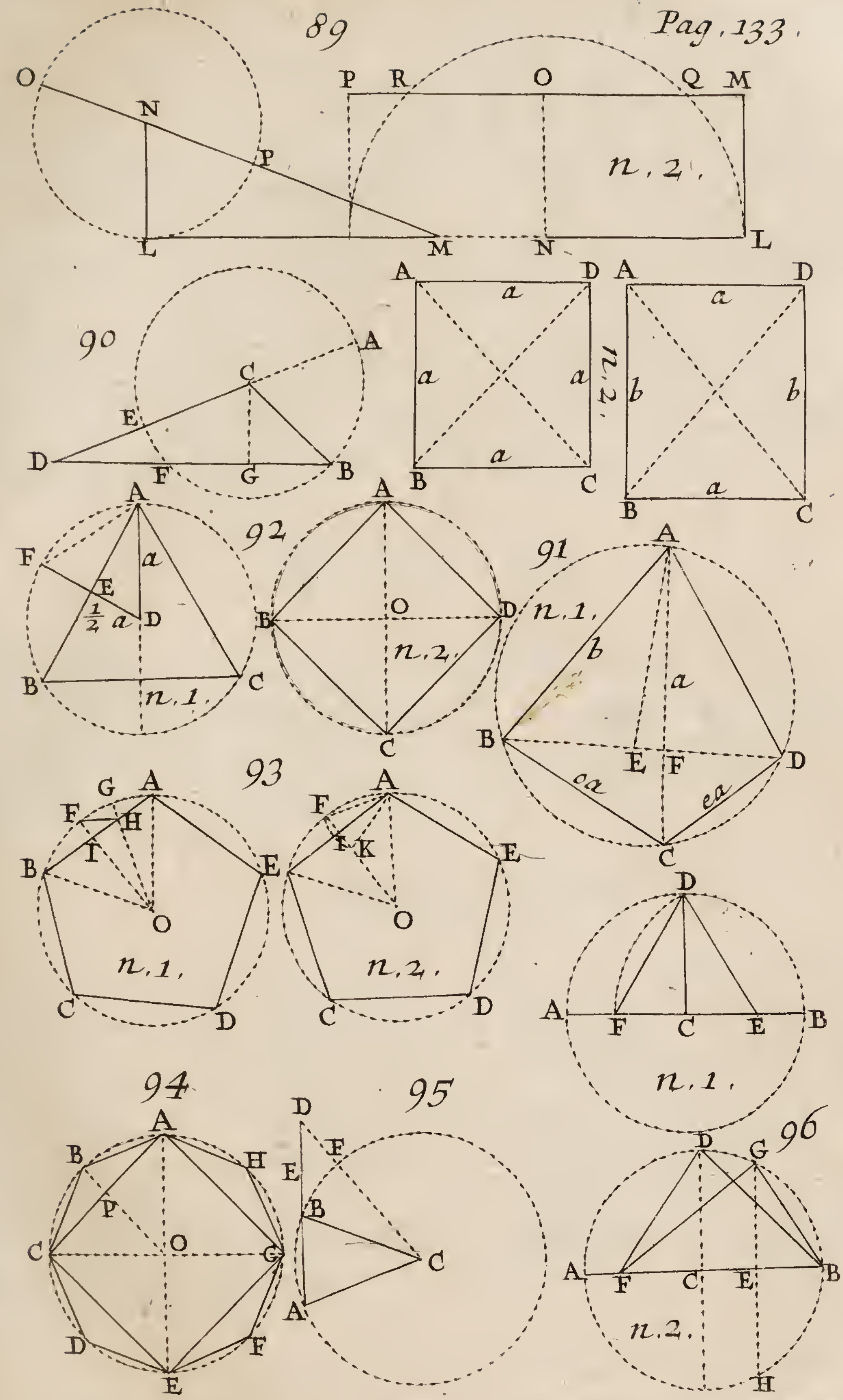
SCHOLIUM I.

TO illuſtrate what we have deduced in the Conſectarys of Prop. 51, you may take the following Notes. If a be put $=10$ or $\frac{1000}{100}$, the ſide of the Decagon will be $=\sqrt{125}-5$, *i.e.* $\frac{618}{100}$ nearly $=b$; therefore the □ $aa=\frac{1000000}{10000}$ and the □ $bb=\frac{381927}{10000}$: therefore $aa+bb$ or the □ $AB=\frac{1381924}{10000}$, the Perpendicular $OI=\frac{a+b}{2}=\frac{1618}{100}$ divided by 2, that is, $\frac{809}{100}$. Now the □ of AI is $\frac{1}{4}$ of the □ $AB=\frac{355481}{10000}$ the □ $OI=\frac{aa+2ab+bb}{4}=\frac{654481}{10000}$. Now if you add the □ AI and the □ OI, the ſum will be $=$ □ $AO=\frac{2aa+2ab+2bb}{4}$, *i.e.* $\frac{aa+2ab+bb}{2}=\frac{99}{1}|\frac{996}{0000}$, or near 100. Thus likewiſe, ſince the Perpendicular above found OI, in *Conſect.* 1. may be alſo determined by $\frac{\sqrt{3aa-bb}}{4}$. ſince aa is $=\frac{1000000}{10000}$, and $3aa\ 3\frac{000000}{10000}$, ſubtracting from it $bb=\frac{381924}{10000}$ the Remainder will be $\frac{2618076}{10000}$, and this being divided by 4, you'l have the □ $OI=\frac{654519}{10000}$. and the Root of it extracted $\frac{809}{100}$ nearly; ſo that thoſe two different quantities in Conſect. 1. will rightly expreſs the ſame Perpendicular OI.

SCHOLIUM II.

NOW therefore as we have Practical Rules to determine Arithmetically the ſides of the Pentagon and Decagon, ſo alſo they may be found Geometrically by what we have demonſtrated. For if the Semidiameter CB (*Fig.* 96. *N.* 1.) be divided into 2 parts, EC will $=\frac{1}{2}a$; and erecting perpendicularly the Radius $CD=a$DE will $=\sqrt{\frac{5}{4}aa}$ Moreover if you cut off EF equal to it, FC will be $=\sqrt{\frac{5}{4}aa}-\frac{1}{2}a=$ to the ſide of the Decagon, by *Conſect.* 1. Having therefore drawn DF, which is equal in Power to the Radius or Side of the Hexagon DC, and the ſide of the Decagon FC together, by the *Pythag. Theorem*

it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it will be the ſide of the Pentagon ſought. Much to the ſame purpoſe is alſo this other new Conſtruction of the ſame Problem, wherein BG (*Numb.* 2.) is the ſide of the Hexagon BD the ſide of the Square, to which GF is made equal, ſo that FC is that ſide of the Decagon, and DF of the Pentagon; which we thus demonſtrate after our way: Having biſected GH the ſide of an Equil. Triangle, the Square of GE will be $\frac{3aa}{4}$, by *Prop.* 48. which being ſubtracted from the Square of GF $= 2aa$, *viz.* $\frac{8}{4}aa$, by *Prop.* 49. there will remain for the Square of EF $\frac{5}{4}aa$, and for the line EF $\sqrt{\frac{5}{4}aa}$, and for FC $\sqrt{\frac{5}{4}aa}-\frac{1}{2}a$, which is the ſide of the Decagon, as DF of the Pentagon, after the ſame way as before.

Propoſition LV.

THE *ſide of a Quindecagon (or* 15 *ſided Figure) is equal in Power to the half Difference between the ſide of the Equil. Triangle and the ſide of the Pentagon, & moreover to the Difference of the Perpendiculars let fall on both ſides taken together.*

Demonſtration.

For if AB (*Fig.* 97.) be the ſide of an Inſcribed Triangle, and De the ſide of a Pentagon parallel to it; AD will be the ſide of the Quindecagon to be inſcribed, by *Conſect.* 4. *Def.* 15. But this ſide AD in the little Right-angled Triangle, is equal in Power to the ſide AH (which is the half Difference between AB and DE) and the ſide HD (which is the difference between the Perpendiculars CF and CG) taken together by the *Pythag. Theor.* Q. E. D.

CON-

CONSECTARY.

HEnce if we call the Side *AB* of the Equil. △ *c*, and make the ſide of the Pentagon DE $= d$, AH will $= \frac{c-d}{2}$ while HD is $= \frac{1}{2} b$, by *Conſect.* 3. of *Prop.* 51. Since therefore the △ AH is $= \frac{cc-2cxdd}{4}$ and the △ H D $= \frac{bb}{4}$ the △ AD will $\frac{cc-2cdxbb}{4}$

Therefore the ſide of the *Quindecagon* $= \sqrt{\frac{cc-2cdxddxbb}{4}}$ that is, Collecting the *S*quare of the half difference of the ſides of the Triangle and Pentagon, and the Square of the difference of the Perpendiculars into one Sum, and then Extracting the *S*quare Root of that Sum, you'l have the ſide of the Quindecagon ſought. *E.* 9. if rhe Radius CI be made 10000000 the difference *of* the ſides of the Triangle 17320508, and of the ſide of the Pentagon 11757704 will be 5564804, and the half of this 2782402; but the difference of the Perpendicular CF from the Perpendicular CG, is 3090170.

The *S*quares therefore of theſe Two laſt Numbers being Collected into one *S*um, nnd the Root Extracted will give the ſide of the Quindecagon 4158234 nearly.

SCHOLIUM.

HEre we will ſhew the Excellent uſe of theſe laſt Propoſitions in making the Tables of Signs. For having found above, ſuppoſing the Radius of 10000000 parts, the ſides of the chief Regular Figures, if they are Biſected, you will have ſo many Primary *S*ines; *viz.* from the ſide of the Triangle the ſide of 60 Degrees 8660754, from the Side of the Square, the Sine of 45° 7071068; from the Side of the Pentagon, the Sine of 36° 5877853; from the Side of the Hexagon, the Sine of 30° 5000000; from the *S*ide of the Octagon, the Sine of 22° 30 3826843; from the Side of the Decagon the ſine of 18° = 3090170; from the ſide of the quindecagon laſtly the ſine of 12° = 2079117. From theſe ſeven primary ſines you may find afterwards the reſt, and conſequently all the Tangents and Secants according to the Rule we have deduc'd, *n.* 3. *Schol.* 5. *Prop.* 34. and which *Ph. Lansbergius* illuſtrates in a prolix Example in his Geom. of Triangles *Lib.* 2 *p.* 7. and the following. But after what way, having found theſe greater numbers of ſines, Tangents, *&c.* Logarithms have been of late accommodated to them, remains now to be ſhewn, which in brief is thus; *viz.* the Logarithms of ſines, *&c.* might immediately be had from the Logarithms of vulgar numbers, if the tables of vulgar numbers were extended ſo far, as to contain ſuch large numbers; and thus the ſine *e. g.* of 0. gr. 34. which is 98900 the Logarithm in the Chiliads of *Vlacquus* is 4995196291б. But becauſe the other ſines which are greater than this are not to be found among vulgar numbers (for they aſcend not beyond 100000, others only reaching to 10000 or 20000) there is a way found of finding the Logarithms of greater numbers, than what are contained in the Tables. *E g.* If the Logarithm of the ſine of 45° which is 7071068 is to be found, now this whole number is not to be found in any vulgar Tables, yet its firſt four notes 7071 are to be found in the vulgar Tables of *Strauchius* with the correſpondent Logarithm 3. 8494808, and the five firſt 70710 in the Tables of *Vlacquus* with the Log. 4. 8494808372. One of theſe Logarithms, *e g.* the latter, is taken out, only by augmenting the Characteriſtick with ſo many units, as there remain notes out of the number propoſed, which are not found in the Tables, ſo that the Log. taken thus out will be 6. 8494808372. Then multiply the remaining notes of the propoſed number by the difference of this Logarithm from the next following, (which for that purpoſe is every where added in the *Vlacquian* Chiliads, and is in this caſe 61419) and from the Product 4176492 caſt away as many notes as adhere to the propoſed number beyond the tabular ones, in this caſe 2; for of the remainder 41764, if they are added to the Logarithm before taken out, there will come the Logarithm required 6. 8494850136, *viz.* according to the Tables of *Vlacquus*, wherein for the Log. of 10 you have 10, 000, 000, 000; but according to thoſe of *Strauchius* which have for the Logarithm of 10 only 10, 000, 000, you muſt cut off the three laſt notes, that the Logarithm of the given ſine may be 6. 8494850; as is found in the *Strauchian* and other tables of ſines, except that inſtead of the Characteriſtick 6 there precedes the Characteriſtick 9, whereof we will add this reaſon: If the Characteriſticks had been kept, as they were found by the rule juſt now given, the Logarithm of the whole ſine (which is in the *Strauchian* Tables 10, 000, 000) would have come out 70, 000, 000, incongruous enough in Trigonometrical Operations. Wherefore that Log. of the whole ſine might begin from 1, for the eaſineſs of Multiplication and Diviſion they have aſſumed 100, 000, 000; the Characteriſtick being augmented by three, wherewith it was conſequently neceſſary to augment alſo all the antecedent ones; and hence *e. g.* the Logarithm of the leaſt ſine 2909 begins from the Characteriſtick 6, which otherwiſe according to the Tables of vulgar numbers would have been 3.

Having found after this way the Logarithms of all the ſines (altho' here alſo if you have found the Logarithms of the ſigns of 45° and moreover the Logarithm of 30, the Logarithms of all the reſt may be compendiouſly found by addition and ſubſtraction from a new principle which now we ſhall omit) the Logarithms of the Tangents and Secants may eaſily be found alſo, only by working; but now Logarithmically, according to the Rules of *Schol.* 5. *Prop.* 34. *n.* 5. and 6.

Propoſition LVI.

(α) Eucl. 13. lib. 13.

THE ſide of a Tetraëdrum (α) *or equil. Pyramid is in power to the Diameter o a circumſcribed Sphere, as* 2 *to* 3.

Demonſtration.

For becauſe by the geneſis of the *Tetraëdrum Def.* 22. (ſee its *Fig.* 44 *n.* 1.) and *Schol. Prop.* 49. OC is $\frac{1}{3}$ of the ſemidiameter OB, which we will call *a* the □ of CB will b $= \frac{8}{9}aa$ by the *Pythag. Theor.* and ſo the power (or Sq.) o the ſide of the *Tetraëdrum* $= \frac{24}{9}aa$ by *Prop.* 49. but the pow

e

er of the Diam. $2a$ or $\frac{6}{3}a$ is $\frac{36}{9}aa$. Therefore the power of the ſide of the *Tetraëdrum* is to the power of the Diam. as 24 to 36. *i. e.* (dividing each ſide by 12) as 2 to 3. Q. E. D.

Or more ſhort.

The $\square$ of CB is $= 2$ by *Schol* of *Prop.* 49 and the $\square$ of EC $= 4$. Therefore the $\square$ EB $= 6$. But the $\square$ EF is $= 9$. Therefore the $\square$ of EB is to the $\square$ of EF as 6 to 9, *i. e.* as 2 to 3. Q. E. D.

CONSECTARY.

THerefore if the Diam. EF be made $= a$, the ſide EB will be $= \sqrt{\frac{2}{3}aa}$.

Propoſition LVII.

THE ſide of the Octaëdrum (α) *is in power one half of the Diameter of the circumſcribed Sphere.*

(α) *Eucl.* 14. *lib.* 13.

Demonſtration.

For ſince by the geneſis of the *Octaëdrum Def* 22. (ſee *Fig.* 44. *n.* 2.) CA, CB, CF, *&c.* are ſo many radii of great circles, if for Radius you put a, the ſquare of AF will be $= 2aa$ by the *Pythag. Theor.* But the ſquare of the Diam. FG $= 2a$ is $4aa$. Therefore the power of the ſide is to the power of the Diam as 2 to 4, *i. e.* as 1 to 2. Q. E. D.

More ſhort.

Becauſe AF is alſo the ſide of a ſquare inſcribed in the greateſt circle by the gen. of the *Octaëdrum*; the $\square$ of AF will be by *Prop.* 50. to the $\square$ of FC as 2 to 1: Therefore to the ſquare of FG as 2 to 4, by *Prop.* 35. Q. E. D.

CONSECTARY.

Therefore if the Diam. of the ſphere be made (a) the ſide of the *Octaëdrum* AF will be $\sqrt{\frac{1}{2}aa}$.

Propoſition LVIII.

(*a*) *Eucl.* 15. *lib.* 13.

The ſide of the Hexaëdrum or Cube (*a*) *is in Power ſubtriple of the diameter of the circumſcribed ſphere.*

Demonſtration.

Making a the ſide of the inſcribed cube GF or FE (*Fig.* 98) the ſquare of the diagonal GE of the baſe of the cube will be $2aa$ by the *Pythag. Theor.* and by the ſame Reaſon the ſquare of the diam. of the cube and the circumſcribed ſphere GD will be $=$ to the ſquare of $GE + \square DE = 3aa$. Q. E. D.

CONSECTARYS.

I. Therefore if the diameter of the ſphere be made $= a$, the ſide of the cube AB will be $\sqrt{\frac{1}{3}aa}$.

II. The diameter of the ſphere is equal in power to the ſide of the *Tetraëdrum* and cube taken together. For if the power of the diam. of the ſphere be made aa the power of the ſide of the *Tetraëdrum* will be $\frac{2}{3}aa$ by *Conſect Prop.* 56. and the power of the ſide of the cube $\frac{1}{3}aa$ by the preſ. *Conſect.* 1. Wherefore theſe two powers jointly make aa. Q. E. D.

Propoſition LIX.

(*a*) *Eucl. Conſect.* 1. *Prop.* 17. *lib.* 13.

The ſide of the Dodecaëdrum (*a*) *is equal in power to the greater part of the ſide of the cube divided in mean and extreme reaſon.*

Demonſtration.

For if to the ſide of the cube AB (*vid. Fig.* 45. *n.* 1.) and to its upper baſe ABCD you conceive to be accommodated or fitted a regular Pentagon according to the geneſis of a *Dodecaëdrum* laid down in *Def.* 22, and at the interval B*e* you make the arch *ef*, the $\triangle$ $\triangle$ AB*e* and A*ef* will be equiangular; (for the angles at A and B being 36°, and A*e*B 108, having drawn *ef*, the angles B*ef* and B*fe* are each 72°; therefore A*ef* the remaining angle will be 36°) wherefore as AB to B*e* (*i. e.* B*f*) ſo A*e* (*i. e.* B*e* or B*f*) to A*f*. Therefore the ſide of the cube AB is divided in mean and extream reaſon in *f*, and B*e* the ſide of the *Dodecaëdrum* is = to the greater part B*f*. Q. E. D.

SCHOLIUM.

HEnce would ariſe a new method of dividing a given line in mean and extream reaſon, *viz.* if you apply to the given line a part of the equilateral Pentagon by means of the angles A and B 36°, and at the interval B*e* you cut off B*f*. This angle may be had geometrically, if another regular Pentagon be inſcribed in a circle, and having drawn alſo a like ſubtenſe, if the angles at the ſubtenſe are made at A and B equal, by *Eucl.* 23. *lib.* 1.

Propoſition LX.

T*HE ſide of an Icoſaëdrum* (α) *is equal in power to the ſide of a Pentagon in a circle containing only five ſides of the Icoſaëdrum; and the ſemi diameter of this circle is in power ſubquintuple of the Diam. of the ſphere of the circumſcribed Icoſaëdrum.*

(α) *Eucl. Prop.* 16. *Coroll. lib.* 13.

Demonſtration.

Both theſe are evident from the geneſis of the *Icoſaëdrum* in *Def.* 22. The firſt immediately hence, becauſe all the other ſides of the triangles (*Fig.* 99.) A*a*, B*a*, *&c.* are made equal to the ſide of the Pentagon AB by *Conſect.* 4. *Prop.* 51. The latter from this inference; if for ☉A the radius of the circle you put *a* (ſince the ſide of the Pentagon, which here is alſo the ſide of the *Icoſaëdrum*, it will be equal in power to the radius and ſide of the Decagon taken together by the aforeſaid *Prop.*) the altitude OG will be the ſide of the Decagon $= \sqrt{\frac{5}{4}aa} - \frac{1}{2}a$ by *Conſect.* 1. *Prop.* 54. to which the equal inferior part *o*H being added, and the intermediate altitude O*o* $= a$, you'l have the whole diameter of the circumſcribed ſphere GH $= a + 2\sqrt{\frac{5}{4}aa} - \frac{2}{2}a$ i. e. $2\sqrt{\frac{5}{4}aa}$ i. e. $\sqrt{\frac{20}{4}aa}$ i.e. $\sqrt{5aa}$ and ſo the ſquare of the diameter of the ſphere will be $5aa$: Therefore the ſquare of the diameter of the ſphere is to the ſquare of the ſemi-diam. of the circle containing the five ſides of the *Icoſaëdrum* as 5 to 1. Q. E. D.

SCHOLIUM.

IT is alſo evident that a ſphere deſcribed on the diameter GH will paſs thro' the other angles of this *Icoſaëdrum*; for aſſuming the center between O and *o* the radius FG will be $= \sqrt{\frac{5}{4}aa}$. But FA is alſo $= \sqrt{\frac{5}{4}aa}$; for the □ of FO is $= \frac{1}{4}aa$, and the □ AO $= aa$: Therefore the ſum is $= \frac{5}{4}aa =$ □ FA. Q. E. D.

CONSECTARY I.

THerefore, if the radius of the circle ABCDE remain *a*, you'l have the altitude OG $\sqrt{\frac{5}{4}aa} - \frac{1}{2}a$, and the ſide of the *Icoſaëdrum* $\sqrt{\frac{5}{2}aa - \sqrt{\frac{5}{4}a^4}}$, by *Conſ.* 1. and 2. *Prop.* 54. and the diam of the circumſcribed Sphere $2\sqrt{\frac{5}{4}aa}$, as is evident from the Demonſtration.

CONSECTARY II.

Being a general one of the five laſt Propoſitions.

(α) *Eucl. Prop.* 18. *and laſt, lib.* 13.

IF AB (*Fig.* 100.) be the diameter of a ſphere (α) divided in D ſo that AD ſhall be $\frac{1}{3}$AB, then (having erected the perpendicular DF) BF will be the ſide of the *Tetraëdrum* by *Prop.* 56. and AF the ſide of the *Hexaëdrum* by *Prop.* 58. *Conſ.* 2. and BE or AE (erecting from the center the perpendicular CE) will be the ſide of the *Octaëdrum* by *Prop.* 57. Now if AF be cut in mean and extreme reaſon in O, you'l have AO the ſide of the *Dodecaëdrum* by *Prop.* 59. Laſtly, if you erect BG double of CB, HI will be double of CI, and the □ of HI = 4 □ of CI; conſequently the □ CH or CB = 5 □ CI: Therefore the □ of AB (double of CH) is alſo = to 5 □ of HI (which is double of CI) therefore HI is the radius of the circle circumſcribing the Pentagon of the *Icoſaëdrum*, and IB the ſide of the *Decagon* inſcribed in the ſame circle, and HB the ſide of the *Pentagon*, and alſo the ſide of the *Icoſaëdrum*, by *Prop.* 60.

The End of the firſt Book.

THE SECOND BOOK.

SECTION I.

Containing

DEFINITIONS.

Definition I.

IF a Cone ABC (*Fig.* 101.) be conceived to be cut by a plane at right angles to the ſide of the cone, *e. g.* BA; the Plane EFGHE ariſing by this ſection, and terminated on the external ſurface of the cone by the curve line HEG, *&c.* was anciently by *Euclid*, *Archimedes*, *&c.* called the *Conick Section*; and if the ſides of the cone AB and BC made a right angle at B, as *n.* 1. the ſection was particularly called the *Section* of a *right-angled Cone*; but if it made an obtuſe angle, as *n.* 2. it was called the *Section of an obtuſe-angled Cone*; if, laſtly, it made an acute one, as *num.* 3. it was called (3.) the *Section of an acute-angled Cone.*

Definition II.

BUT afterwards their Succeſſors, particularly *Apollonius Pergæus*, found from the properties of theſe Curves, which their Predeceſſors had happily diſcovered, that the ſame (all of them) might be generated in one and the ſame cone whether right-angled, obtuſe-angled, or acute-angled, if the ſection EF (*Fig.* 102.) is made in the firſt caſe parallel to the oppoſite ſide BC; in the ſecond caſe, if it meet that ſide produced upwards; for the third, when it meets downwards, with the diameter of the baſe AC produced to D. And to give new names (for the old ones would do no more now) to theſe Sections, to diſtinguiſh them one from another, nominating them from their Properties hereafter demonſtrated, they called the firſt a *Parabola*, the ſecond an *Hyperbola*, the third an *Ellipſis.*

Definition III.

BUT it is evident, that only the plane making the ſection of the ſecond caſe, being according to the line FED produced or carried on, (*Fig.* 103.) will fall upon the vertical Cone *a*B*c* comprehended under the ſides AB, CB, *&c.* continued onwards, and there produce another Section oppoſite to the former; whence theſe, *viz.* GEHG and *gehg* are called *oppoſite Sections.*

Definition IV.

BEſides theſe names of the ſections, there are ſeveral others made uſe of to denote various lines drawn and conſidered both within and without thoſe ſections, the chief whereof we ſhall here explain. And firſt of all, in general the line EF ſo let fall thro' the middle of the ſection from its top E (which is called the *Vertex* of the ſection) to the diameter of the baſe of the Cone AC (produced if occaſion be) that it ſhall biſect the line GH and all others parallel to it, is called the *Diameter* of that *Section*; and particularly it is called the *Axis* of the *Section* if it cuts them at right angles or perpendicularly; as alſo they name thoſe lines GH, KN, *&c.* which are cut indifferently by the diameters, but at right angles by the Axis, thoſe, I ſay, they call *Ordinates*, or *Ordinate Applicates*, and their halves, FG, IK, *&c. Semiordinates*, (or ſome alſo call the latter *Ordinates*, and the former *double Ordinates*) and the parts of the Ax or Diameter EF, EI, *&c.* are called *Abſciſſa's* (by ſome *intercepted* axes and diameters)

Definition V.

PArticularly in the hyperbola they call the continuation of the ax or diameter ED 'till it meet the oppoſite ſide *c*B, *i. e.* to the vertex of the oppoſite ſection, the *Latus tranſverſum* of the Hyperbola, to which there anſwers in the Ellipſis the axis or longeſt diameter, and ſo by latter Authors is called by the ſame name, but by *Apollonius* the *tranſverſe Ax* or the *tranſverſe Diameter*, as alſo the ſhorteſt ax or diameter is called the *Conjugate*, and its middle point O is called the *Center of the Section* or of the *oppoſite Sections.*

Definition VI.

THey called alſo a certain line EL (*Fig.* 101.) by the name of *Latus Rectum*, which is particularly to be found in all the ſections, as we ſhall hereafter ſhew: And becauſe this *Latus Rectum* is a ſort of a Rule or Meaſure, according to which the ſquares or powers of the ordinates uſed to be eſtimated or valued (as we will ſhew in its proper place) therefore the Ancients uſed to call it by a Periphraſis *Linea ſecundum quam poſſunt Ordinatim applicatæ*, or the meaſure of the powers of the Ordinates; by ſome latter Authors it is called the *Parameter.* Now a mean proportional PQ found between this *Latus Rectum* and the *Latus Tranſverſum* (*Fig.* 104 *n.* 1. and 2) (ſee alſo hereafter *Conſect.* 2. *Prop.* 8.) and drawn thro' the centre O parallel to the Ordinates is called the *ſecond Axis* or *Diameter*, or the *Conjugate* of the Hyperbola.

Definition VII.

NOW if we conceive the diameter or conjugate ax PQ brought down to the Hyperbola ſo that its middle point O ſhall touch its vertex in E, and from the center O you draw the right lines OR, OS, thro' the ends of this tangent line *p* and *q*, theſe are the lines which *Apollonius, Prop.* 1. *lib.* 2. demonſtrates, that tho' by being continued, they always approach nearer and nearer to the curve GEH, and come ſo much

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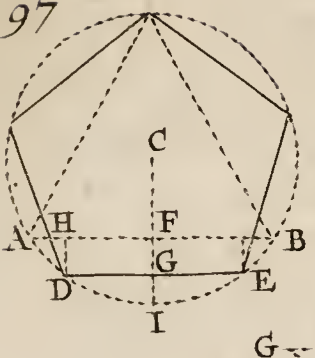

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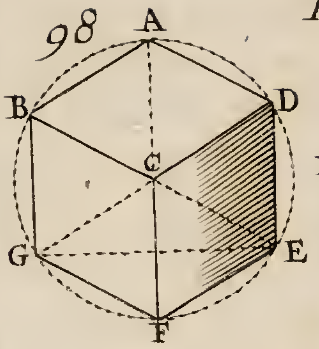

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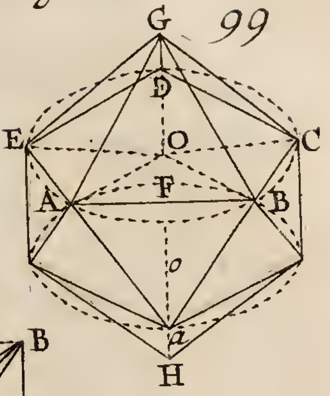

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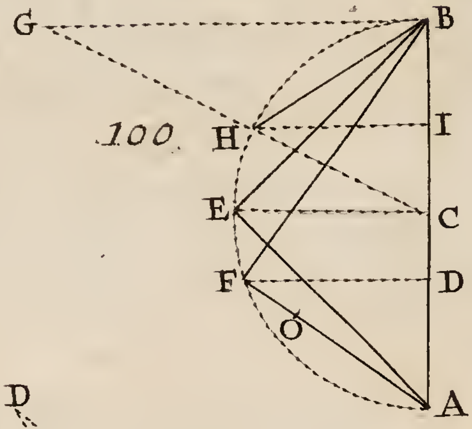

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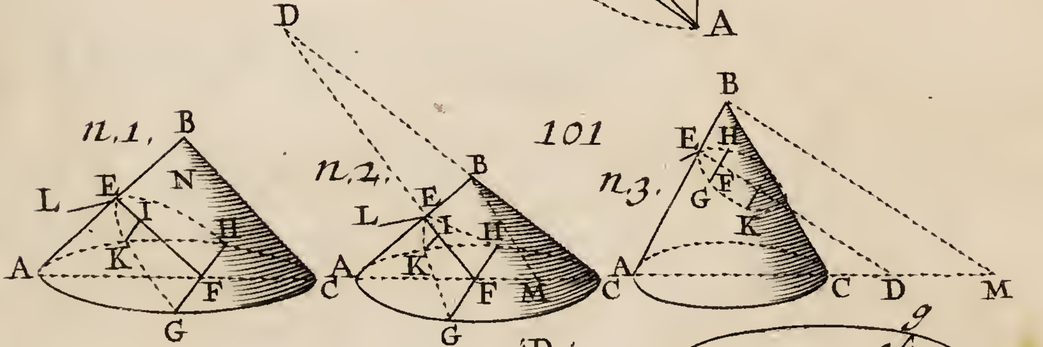

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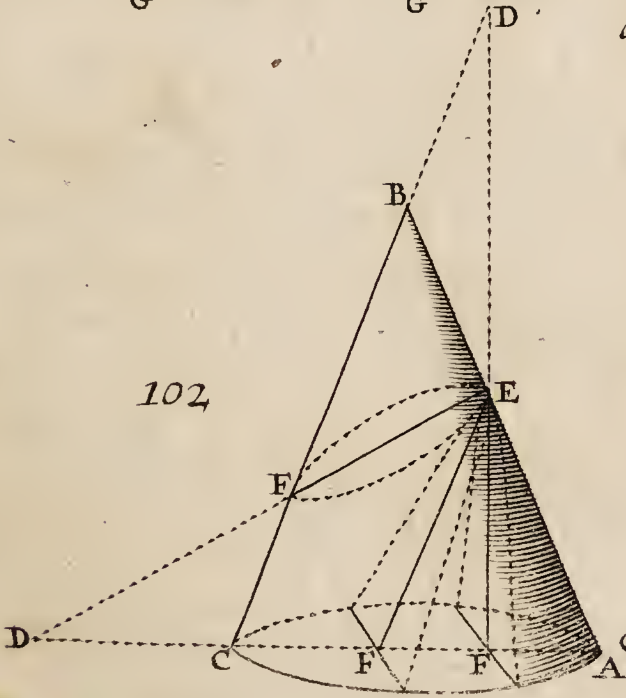

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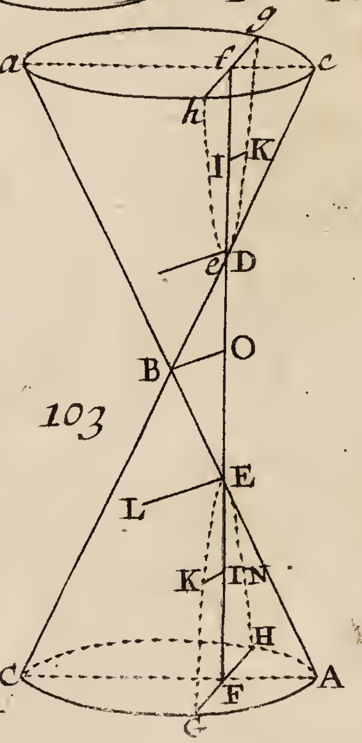


much the nearer by how much the farther they are continued, yet they will never concur with it or touch it, for which reaſon they are called Aſymptotes or non-coincident lines; and by ſome the *Intactæ*. Which non-coincidency appears moſt manifeſtly where the hyperbolical ſection of the cone is made parallel to the triangular ſection thro' the axis of the cone ABC (*n.* 3.) along the line *e. f.* parallel to the ax BF. For if we conceive the hyperbola *geh* to move forwards always parallel to it ſelf, according to the direction of the equal and parallel lines *g*G, *f*F, and *h*H, 'till it ſtands in the poſition GEH; it is manifeſt that the curve line GEH is diſtant on both ſides from the right lines BC, BA, the length of the verſed ſine of the equal arches *h*C and *g*A in the circumference of the circular baſe, while in the mean time it is evident that they approach nearer and nearer to them. So that hence there flow the following

CONSECTARYS.

I. IN this caſe the ſides of the cone are the Aſymptotes of the hyperbola, while it is manifeſt, that the point B is its centre, and EB half the tranſverſe diameter; which appears from *n* 1. and 2. of the pref. *Fig.* for the ſection *ef* being made parallel to the the ax of the cone DF by *def.* 5. *d e* (which in the caſe *n.* 3. would coincide with *dq*) is the tranſverſe diameter, but the triangles *dpq* and O*p*E are equiangular, and conſequently as *p*E is to $\frac{1}{2}pq$, ſo is OE to $\frac{1}{2}dq$.

II. The lines AG and HC (*num.* 3) are equal, as being the verſed ſines of equal arches; and in like manner (*n.* 1. and 2.) RG and HS are equal, ſince FR and FS as well as E*p* and E*q* are ſo alſo (for

$$\left.\begin{array}{l}\text{as OE to E}p\text{, ſo OF to FR}\\ \qquad\text{—E}q \qquad\qquad \text{—FS}\end{array}\right\}$$ and the ſemiordinates FG and FH are alſo equal.

III. Conſequently ▭ ▭ of RG into GS and of HS into HR are equal, *&c.* all which hereafter we will more univerſally demonſtrate.

Definition VIII.

(α) Archim. de Conoid. & Spher. Def. 1.

IF a parabolick plane (α) HEGFH (*Fig.* 105. *n.* 1.) together with a triangle HEG inſcribed in it, and a rectangle HL circumſcribed about it, be conceived to be moved round about the ax EF on the point F; it will be evident that by the triangle there will be generated a cone, by the rectangle a cylinder, and by the parabola a parabolick ſolid, which with the comprehended cone, and the comprehending cylinder, will have the common baſe HIGK and the ſame altitude EF, and was by *Archimedes* named a *Parabolick Conoid.*

Definition IX.

(α) Archim. de Conoid. & Spher. Def. 3.

IF moreover an (α) hyperbolick plane HEGFH (*n.* 2.) with the inſcribed triangle HEG, and another circumſcribed one ROS made by the Aſymptotes OR, OS, be conceived to be turned about the common ax OEF on the point F; it will be evident that there will be deſcribed by the inſcribed triangle a cone comprehended within ſide, and by the hyperbola an *Hyperbolical Conoid* upon the ſame baſe HIGK and of the ſame heighth EF; and by the △ ROS another cone which *Archimedes* calls the comprehending cone, whoſe baſe is RTSV, and its altitude compoſed of the axis of the hyperbola EF and half the tranſverſe ax OE (which *Archimedes* called the additament of the ax of the hyperbola) and which we may commodiouſly divide into two parts, *viz.* into the cone OPMQL, whoſe baſe has for its diameter the conjugate ax PQ, and its altitude equal to half the tranſverſe ax; and into a *Curti-cone* or detruncated cone terminated by the two baſes PMQL and RTSV, but anſwering in altitude to the conoid and inſcribed cone: From which, as comprehending it, if you take away the included conoid, there will remain the hollow curticone terminated below by the *Annulus* or ring RGIVHSKT and above by the circular baſe PMQL, and generated in the circumvolution by the intermediate lines EP, GR, *&c.* or the mixtilinear plane EGRP. Now if we ſuppoſe inſtead of this comprehending cone a circumſcribed cylinder on the ſame baſe and of the ſame altitude with the conoid and included cone, you'l have every thing like as in *Def.* 8.

CONSECTARYS.

NOW if we ſuppoſe the 1. caſe of *Def.* 9. to be expreſſed by the *Fig.* of *Def.* 7. *n.* 3. and conceive the preſent figure brought thence to be turned round about the ax BEF (*Fig.* 106.) we may deduce theſe following things in the room of Conſectarys for the foundation of our future demonſtrations.

I. The lines EQ, RS, HC, *&c.* of the mixtilinear ſpace comprehended between the hyperbola and the Aſymptotes (*viz.* the exceſs of the ordinates) altho' they are unequal, and by deſcending always grow leſs, yet in this circumvolution they will deſcribe equal circular ſpaces, *viz.* EQ a whole circle, (or circular plane) but RS and HC, *&c.* circular *Annuli* or rings all of the ſame bigneſs; which will thus appear to any one who compares this figure with the former: Since the ſpaces generated by the lines EQ, FC, *&c.* are as the ſquares of thoſe lines, and the □ of F*h* or FC exceeds the ſquare of *f h* or FH by the exceſs of the ſquare F*f* or E*e* or EQ, conſequently the quantity generated by FC will exceed that generated by FH the exceſs of that generated by EQ; and alſo that generated by FH by the exceſs of that generated by HC; it is manifeſt that the circle generated by EQ will be equal to the annular ſpace generated by HC; and the ſame will in like manner be evident of any ſpaces produced by RS.

II. Therefore the hollow detruncated cone generated by the ſpace EHCQ according to *Def.* 9. will be equal to a cylinder generated by the rectangle FIQE; for all the indiviſibles of the one, are equal to all the indiviſibles of the other by *Conſect.* 1.

Defi-

Definition X.

(*a*) *Archim. de Conoid & Spher. Def.* 6.

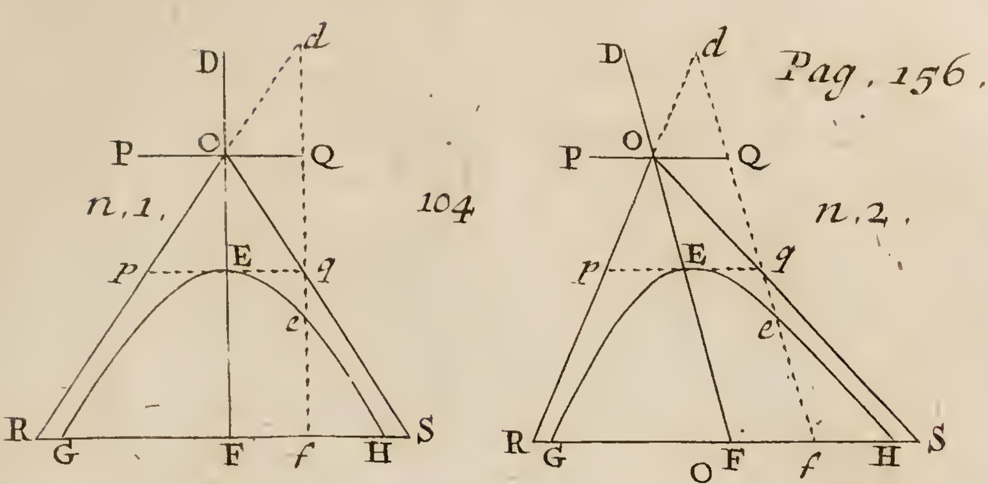
IF, laſtly, (*a*) an elliptical plane be turned about either of the axes, *viz.* either the longeſt DE (*Fig.* 107. *n.* 1.) or the ſhorter AB (*n.* 2.) there will be thence formed an elliptical ſolid, called by *Archimedes* a ſpheroid; which in the firſt caſe will be an oblong or erect one, in the other a flat or depreſſed one: And it is ſelf evident, that if before this circumvolution of the ellipſis, there be inſcribed in one of its halves a triangle, and alſo a rectangle circumſcribed about it, having the ſame altitudes and baſes with the ſemi ellipſe, there will afterwards in the circumvolution be deſcribed by the triangle a cone, by the rectangle a cylinder, to which afterwards we will alſo compare the half ſpheroid; as alſo both the conoids with their inſcribed and circumſcribed cones and cylinders.

Definition XI.

IF upon the right line AE (*Fig.* 18.) you conceive a wheel or circle to rowl, until its point A, with which it touches the ſaid line, come to touch it again in E; the circle wil meaſure out the line AE equal to its periphery; but the point A by its motion will deſcribe the curve line AFE, which is called a *Trochoid* or *Cycloid*; and the area which this curve with the ſubtenſe AE comprehends, is named the *Cycloidal Space*; and the circle by whoſe motion they are determined is called the *generating Circle.*

CONSECTARY.

IT is evident from the geneſis of this curve that the deſcribing point *a* will always be as much diſtant in the circle from the point of contact *d* or *c*, as the point A in the right line paſſed over AE, is from the ſame point of contact, *i. e.* if the point *d* is diſtant from A the fourth part of the line AE, the arch *da* will alſo be the fourth part of the circle conſidered in this ſecond ſtation; and the point *c* being diſtant from A half of the interval AE, the arch *ca* will be alſo half of the circle,



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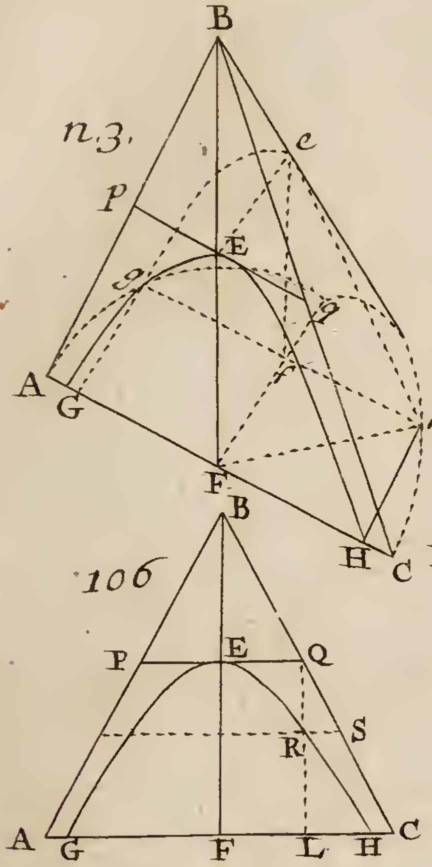

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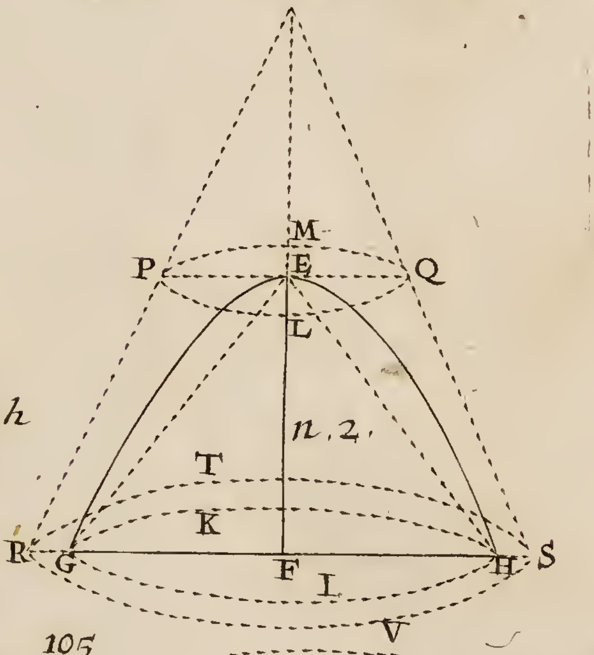

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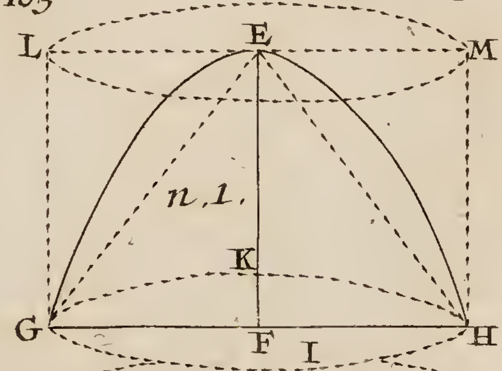

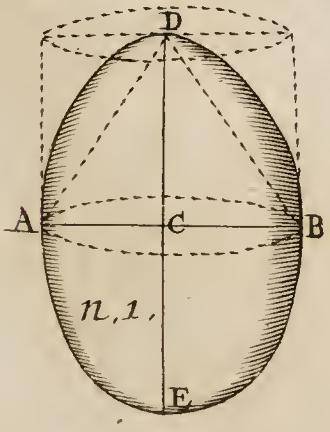

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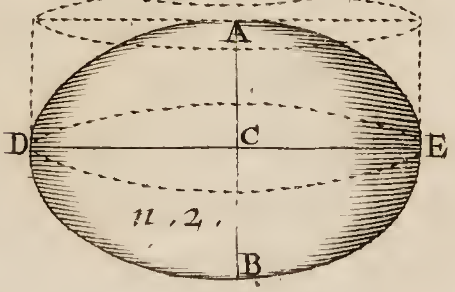

circle, and ſo the point *a* coincide in the curve with F: And when the point *e* is diſtant from E only an eighth part of the whole line AE, the arch *ea* will alſo be the eighth part of the whole circle.

Definition XII.

If the right line BA (*num.* 1. 109.) one end at B remaining fixed, be moved round at the other end with an equal motion from A thro' C, D, E to A back again, and in the mean while, there be conceived another moveable point in it to move with an equal motion along the line BA from B to A, ſo that in the ſame moment wherein the moveable point A abſolves one revolution, the other moveable point ſhall alſo have paſſed thro' its right lined way, coinciding with the point A returned to its firſt ſituation; this extremity A by its revolution will deſcribe the circle ACDEA, and that other moveable point another curve B, 1, 2, 3, 4, *&c.* which with *Archimedes* we will call a *Helix* or *ſpiral Line*, and the plane ſpace comprehended under this ſpiral line and the right line BA in the firſt ſtation is called a *ſpiral ſpace.* Now if we ſuppoſe, *e.g.* the right lined motion of the point moving along BA to be twice ſlower than in the former caſe, ſo that (ſee *num.* 2.) in the ſame time that the point A makes one whole Revolution, the other moveable point ſhall come to F, making half the way BA, and then at length ſhall concur or meet with the extremity, when that ſhall have performed the other revolution; and ſo there will be deſcribed a double ſpiral line, whoſe parts with *Archimedes* we will ſo diſtinguiſh, that as he calls the part of the right line BF, paſſed over in the firſt revolution, ſimply the *firſt line*, and the circle made by the right line BF the *firſt Circle*; ſo we will call that part of the curve which is deſcribed in that time or revolution B 1 3 6 9 12 the *firſt Helix* or the *firſt Spiral*, and the area comprehended under it the *firſt ſpiral ſpace*: And, as the other part of the right line FA paſſed over in the other revolution is called the *ſecond line*, and the circle marked out by the whole line BA the *ſecond Circle*; ſo the curve deſcribed in the mean while 12, 15, 18, 24, may be called the *ſecond ſpiral line*, and the ſpace comprehended under it the *ſecond ſpiral ſpace*, and ſo onwards From theſe Definitions there flow the following

CONSECTARYS.

I. THE lines B 12, B 11, B 10, *&c.* drawn out, and making equal angles to the firſt or ſecond ſpiral (and after the ſame manner *(α)* B 12, B 10, B 8, *&c.* or B 12, B 9, B 6, *&c.)* are arithmetically proportional,as is evident.

(α) *Archim. Prop.* 2. *of Spirals.*

II. The lines drawn out to the firſt ſpiral as B 7, B 10, *&c.* are one among another as the arches of the circles intercepted between BA and the ſaid lines *(β)* B 7, B 10, *&c.* which is alſo evident to any one who conſiders what we did ſuppoſe; for in the ſame time as the end A paſſes over ſeven parts of the circle, the other moveable point will alſo run over ſeven parts of the right line BA, *&c.*

(β) *Archim. Prop.* 4.

III. Laſtly, The right lines drawn from the initial point *(γ)* B to the ſecond ſpiral *e. g.* B 19 and B 22 *(num.* 2.*)* will be one to another as the aforeſaid arches together with the whole periphery added to both ſides: for at the ſame time the extremity A runs thro' the whole circle or 12 parts and moreover 7 parts (*i. e.* in all 19 parts*)* in the ſame time the other moveable point paſſes through 12 parts of the right line BA (in this caſe divided into 24 parts*)* and moreover 7 parts, that is, in all 19; and ſo in the others.

(γ) *Archim. Prop.* 15.

Definition XIII.

IF a right line BAF be conceived ſo to move within the right angles ADC, CDE, that on the one hand a certain point C fixed in one leg of the *Norma* or ruler may always glide along, and on the other hand a certain moveable point A may always run along the other ſide of the *Norma* (which complicated motion of the deſcribing rule BAF, after what way it may be organically procured, may be ſeen by the

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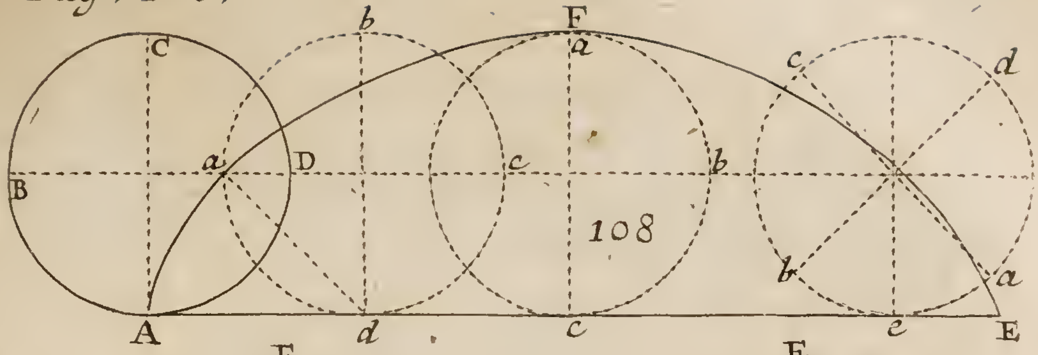

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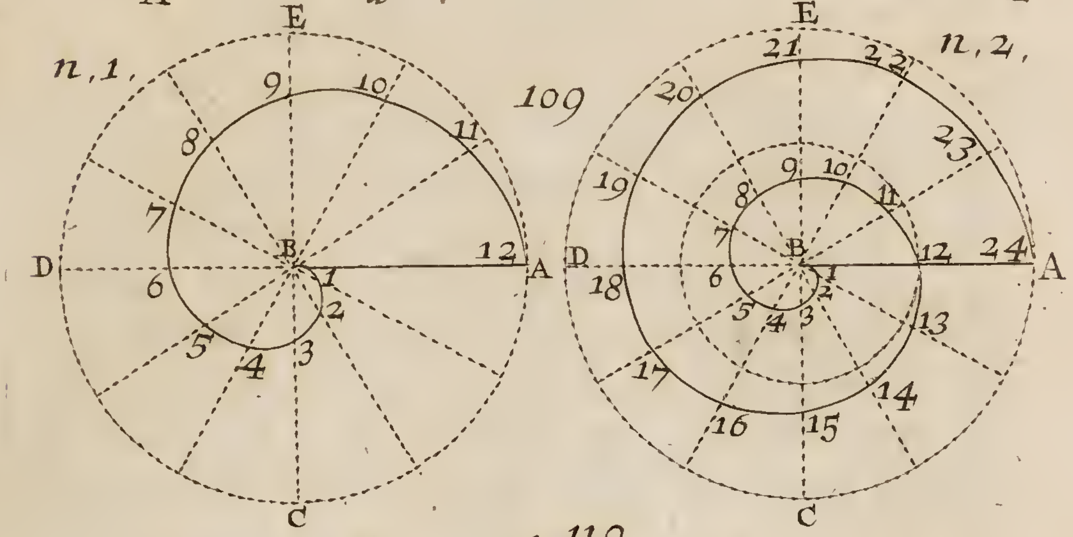

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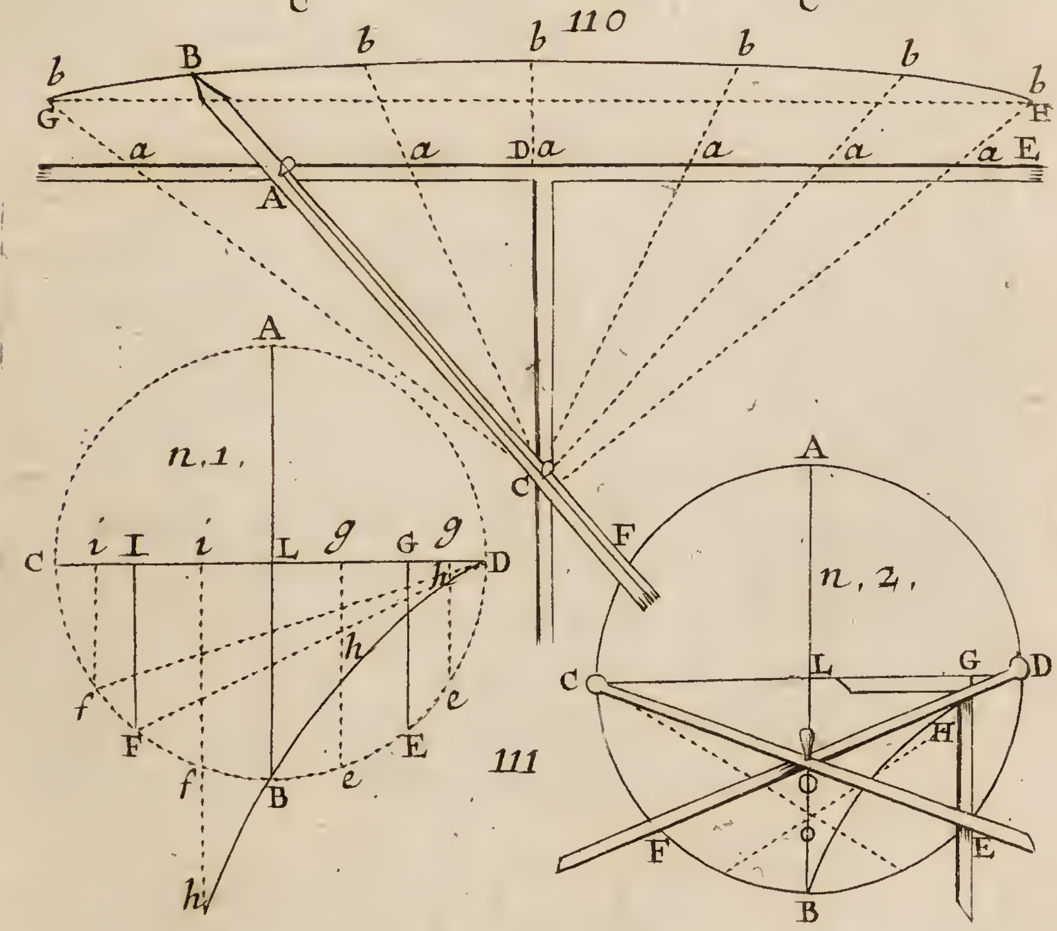

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111

the 110*th.* Figure;) by the extreme point of the moveable line B there will be deſcribed a curve called by its Inventor *Nicomedes* the *firſt Conchoid*, whereof this is a property, that the right lines CB, C*b*, drawn from its centre C to its *Ambitus* or curvity are not themſelves, as in the circle, equal, but yet have all the parts, AB, *ab*, intercepted between the curve, and the directrix AE are equal; as is evident from its geneſis.

Definition XIV.

IF, the diameters of a circle being AB and CD (*Fig.* 111. *n.* 1.) cutting one another at right angles, you take BE or B*e* and BF or B*f* equal arches, and from E or *e* you draw the perpendiculars EG or E*g*, and through theſe from D to F or *f* you draw the tranſverſe lines DF or D*f*; the ſeveral points of interſection H, *h*, *&c.* decently connected together will exhibit the curve line D*h*, H*h*B (which may be continued alſo without the circle if you pleaſe, and) which is commonly attributed to *Diocles*, and called a *Ciſſoid.*

Definition XV.

IF, having divided the right line AE (*Fig.* 112.) into the equal parts AB, BC, *&c.* from the points of the diviſion A, B, C, *&c.* you draw the parallel lines A*a*, B*b*, C*c*, *&c.* in geometrical progreſſion (as *e.g.* let A*a* be 1, B*b* 4, C*c* 16, D*d*, 64, *&c.* or B*b* 10, C*c* 100, D*d* 1000, E*e* 10000, *&c.*) and further biſecting AB, BC, *&c*, 𝔉, 𝔊, ℌ, ℑ, you let fall mean proportionals between the next Collaterals 𝔉𝔣, 𝔊𝔤, ℌ𝔥, ℑ𝔦, *&c.* and continue to do ſo till the parallels are brought near to one another; the curve line drawn thro' the extremities of theſe lines *a*𝔣, *b*𝔤, *c*𝔥, *d*𝔦𝔢, will be the logarithmical line of the moderns, whoſe properties and uſes are very excellent.

SCHOLIUM.

AMong thoſe uſes, that is none of the leaſt, from which this curve borrows its name, *viz.* in ſhewing the nature and invention of Logarithms. For, *e g.* 1. If this line was accurately delineated in a large ſpace, the parts AB, BC, *&c.* being taken ſo big, that they might be ſubdivided not only into 100 or 1000 but into 10000 or 100000 parts; making AB 100000 (and ſo A 00000) AC would be 200000, AD 300000, *&c.* while in the mean while there anſwer to theſe as primary Logarithms in arithmetical progreſſion the geometrical proportional numbers, A*a* 1, B*b* 10, C*c* 100, D*d* 1000, E*e* 10000, *&c.* Whence, 2. Its Logarithm may be aſſign'd to any given intermediate number, *e.g.* to the number 982, for having cut off this number from D*d* by a geometrical ſcale on the line DM, if you draw M*n* parallel to AD, and *n*N parallel to DM, it will give AN on the ſame ſcale, *viz.* the Logarithm ſought, and reciprocally. But if, 3. it ſeem difficult to delineate a Figure ſo large, yet at leaſt the clear conception of ſuch a delineation evidently ſhews the arithmetical method, which thoſe ingenious Men have made uſe of, who have made the tables of Logarithms with a great expence of Labour and pains, *viz.* by finding continual mean proportionals, arithmetical ones between any two Logarithms already known, and geometrical ones between two vulgar numbers anſwering to them, *&c.* by comparing what we have noted in *Schol.* 2. *Prop.* 20. *Lib.* 1. And we will note, 4 out of *Pardies*, that, ſince the Logarithms of numbers diſtant from one another by a decuple proportion, differ by the number 100000, having found the Logarithms of all the numbers from 1000 to 10000 you will at the ſame time have all the Logarithms of all the other numbers that are between 100 and 1000, between 10 and 100, and between 1 and 10, only changing the characteriſtick, and leſſening it in the firſt caſe by unity, in the ſecond by 2, in the third by 3; as *e.g.* if of the number 9, 900 you had found the Logarithm 399, 563, the Logarithm of the ſubdecuple number 990 would be (*viz.* ſubſtracting from the former 100000) 299, 563. and the Logarithm again of this ſubdecuple of this 99 would be 199, 563, *&c.* Thus in the Chiliads of *Briggs* to the number

99000	Anſw.	4, 99563, 51946
9900	——	3, 99563, 51946
990	——	2, 99563, 51946
99	laſtly ——	1, 99563, 51946

But there will not ariſe ſuch advantage for making Logarithms by this obſervation as it may at firſt ſight ſeem to promiſe, becauſe there are 9000 numbers between 1000 and 10000 whoſe Logarithms muſt be found alſo, and but 900 between 100 and 1000, and but 90 between 10 and 100, and but 9 between 1 and 10, and ſo in all 999, which is not the ninth part of the former.

Definition XVI.

IF the radius AD (*Fig.* 113.) be conceived to move equally about the point A through the periphery of the quadrant DB, while in the mean time the ſide of the ſquare DC remaining always parallel to it ſelf, deſcends alſo with an equal motion thro' DA, ſo that in the ſame moment the radius AD and the aforeſaid ſide DC ſhall fall upon the baſe AB; or (if any one ſhould think that this way the proportion of a right line to a circular one is ſuppoſed by a ſort of *Petitio Principii* or begging the queſtion) the right line DA as well as the quadrant DB being divided into as many equal parts as you pleaſe (*e. g* here both of them into 8) and drawing thro' theſe from the center A ſo many Radii and thro' them parallel lines; the points of interſection being orderly connected together will exhibit a curve line, whoſe invention is attributed to *Dinoſtratus* and *Nicomedes* in the fourth Book of *Pappus Alexandrinus*, and which from its uſe is called a *Quadratrix*, it having among the reſt this property, that from AB it cuts off a part AE, which is a third proportional to the quadrant DB and its radius DA; which hereafter we will demonſtrate. In the mean while from this deſcription of it, you have theſe

CONSECTARYS.

I. IF thro' any point H aſſumed in the *Quadratrix* you draw the radius AHI, and from the ſame point the perpendiculars H*b* and H*e*, the whole arch DB will always be to the part IB cut off, as the whole line DA to the part *b*A cut off, or H*e* equal to it.

II. Conſequently therefore any given arch or angle of the quadrant *e. g.* IB or IAB may by help of the quadratrix be divided into three equal parts or as many as you pleaſe, or in what proportion ſoever you will; while having drawn the radius AI, the perpendicular H*a* let fall from the point of the quadratrix H, may be divided into three or as many equal parts as you pleaſe, or in any proportion whatſoever, and thro' theſe ſections radius's drawn to divide the arch.

BOOK II.

SECTION II.

CHAP. I.

Of the chief Properties of the Conick Sections.

Propoſition I.

(α) I. *Property of the Parab. Apoll. Prop.* 11. *Lib.* 1.

IN the Parabola (GKEH Fig. 114.) *the* (α) *ſquare of the ſemi-ordinate (IK) is equal to the rectangle IL made by the Latus Rectum EL and the abſciſſa EI.*

Demonſtration.

MAke the ſides of the cone that is ſuppoſed to be cut, AB = *a*, BC = *b*, and moreover EB = *oa*, and EI = *eb*, and AC = *c*; therefore NI will be = *ec*, by reaſon of the ſimilitude of the △ △ BCA and EIN; and EP or IO = *oc*, by reaſon of the ſimilitude of the △ △ ABC and EBP. Therefore ▭ NIO = *oecc* = □ IK by the *Schol.* of *Prop.* 34. (*n.* 3) and *Prop* 17. *Lib.* 1. Now if a line be ſought which with the abſciſſa EI ſhall make the ▭ IL = □ IK you will have it by dividing the ſaid ſquare by the Abſciſſa EI. *viz.* $\frac{oecc}{eb}$ i. e.. $\frac{occ}{b}$ ⌶ EL. And this is called the *Latus Rectum, viz.* in relation to the Abſciſſa EI with which it makes that rectangle, which, it's evident, is ⌶ □ IK, and from this equality the ſection has the name of Parabola, in *Apollonius.*

CONSECTARYS.

I. THis *Latus Rectum,* expreſſed by the quantity $\frac{occ}{b}$, may be found out after a ſhorter way, if you make as *b* to *c* (the ſide of the cone parallel to the ſection BC at the Diameter of the baſe AC) ſo *oc* (the ſide EP called by ſome the *Latus Primarium)* to a fourth.

II. But if any one, with *Apollonius,* had rather expreſs this by meer *data* in the cone it ſelf as cut (becauſe *oc* or that *Latus Primarium* EP is not a line belonging to the cone it ſelf) he may eaſily perceive, if the quantity of the *Latus Rectum* found above, be multiplyed by the other ſide of the cone *a,* there will be produc'd the equivalent $\frac{oacc}{ab}$ which inſtead of the proportion above will furniſh us with this other,

$$\left.\begin{array}{l} \text{as } ab \text{ —— to } cc \text{ —— ſo } oa \\ \square \text{ of AB into BC — } \square \text{ AC — EB} \end{array}\right\} \text{to a fourth ;}$$

which is the very proportion of *Apollonius* in *Prop.* 9. *Lib.* 1. and confirms our former.

SCHOLIUM I.

HEnce you have an eaſie and plain way of deſcribing a Parabola, having the top of the ax and the *Latus Rectum* given, *viz.* by drawing ſeveral ſemiordinates whoſe extreme points connected together will exhibit the curvity of the Parabola. But you may find as many ſemiordinates as you pleaſe, if having cut off as many parts of the Ax as you pleaſe, you find as many mean proportionals between the

Latus Rectum and each of thoſe parts or Abſciſſa's. See *n.* 2. and 3. *Fig.* 47. *Introduct. to Specious Analyſis.*

SCHOLIUM II.

HEnce alſo we have a new geneſis of the parabola in *Plano* from the ſpeculations of *De Witte*, *viz.* if the rectilinear angle HBG (*Fig.* 115.) conceived to be moveable about the fixed point B be conceived ſo to move out of its firſt ſituation with its other leg BH along the immoveable rule EF, that it may at the ſame time move alſo the ruler HG, from its firſt ſituation DK, all along parallel to it ſelf, and with the other leg BG let it all along cut the ſaid ruler HG, and with this point of its interſection continually moving from B towards G it will deſcribe a curve. That this curve will be the parabola of the antients is hence manifeſt, becauſe it will have this ſame firſt property of the parabola. For, 1. if the angle HBG (*n.* 1.) be ſuppoſed to be a right one, and BD or HI $= a$, BI or KG $= b$ (*viz.* in that ſtation of the angle and rule HG by which they denote the point G in the interſection) you'l have by reaſon of the right angle at B, BI, *i. e.* b a mean proportional Between HI *i. e.* a, and IG or BK, and ſo this as an abſciſſa $= \frac{bb}{a}$. Wherefore if BK *i. e.* $\frac{bb}{a}$ be multiplyed by BD $= a$, the rectangle DBK will be $= bb = \square$ KG; which is the firſt property of the parabola: So that it follows, ſince the ſame inference may be made of any other point in this curve, that this curve will be the parabola, BD or HI its *Latus Rectum*, KG a ſemiordinate, and BK its axis, *&c.* 2. If the angle HBG be an oblique one (*num.* 2) it may be eaſily ſhewn from what we have ſuppoſed that the $\triangle\ \triangle$ DBH and BKG will be equiangular: Therefore as BD (i. e. a) to DH ſc. BI (i. e. b) ſo KG ſc BI (i. e. b) to BK (i. e. $\frac{bb}{a}$.) Therefore again the $\square$ DBK $= bb\ \square$ KG. Q. E D.

Conſect. 3. It is alſo evident in this ſecond caſe, that BK drawn parallel to the ax, but not thro' the middle of the parabola, will be a diameter which will have for its vertex B, its *Latus Rectum* BD, and ſemiordinate GK, *&c.*

Conſect. 4. Therefore you may find the *Latus Rectum* in a given parabola geometrically, if you draw any ſemiordinate whatſoever IK (*Fig.* 116.) and make the abſciſſa EF equal to it, and from F draw a parallel to the ſemiordinate IK, and from E draw the right line EK thro' K cutting off FH the *Latus Rectum* ſought; ſince as EI to IK ſo is EF (*i. e.* IK) to FH by *Prop* 34. *lib.* 1. wherefore having the abſciſſa and ſemiordinate given arithmetically, the *Latus Rectum* will be a third proportional.

Conſect. 5. Since therefore the *Latus Rectum* found above is $\frac{occ}{b}$, if you conceive it to be applyed to the parabola in LM, ſo that N ſhall be that point which is called the *Focus*, LN will be $\frac{occ}{2b}$ and its ſquare $\frac{ooc^4}{4bb}$ and this divided by the *Latus Rectum* $\frac{occ}{b}$ will give $\frac{occ}{4b}$ for the abſciſſa EN: So that the *diſtance of the Focus from the Vertex* will be $\frac{1}{4}$ of the *Latus Rectum.*

Conſect. 6. Since therefore EN is $= \frac{occ}{4b}$, if for EF you put ib, NF will be $= \frac{ib - occ}{4b}$, whoſe ſquare will be found to be $i^2b^2 - \frac{oicc}{2} + \frac{o^2c^4}{16b^2}$ to which if there be added $\square GF = oicc$, by *Prop.* 1. the ſquare of NG will be $= i^2b^2 + \frac{o1cc}{2} + \frac{o2c^4}{16b^2}$ whoſe root (as the extraction of it and, without that, the analogy of the ſquare NF with the ſquare NG manifeſtly ſhews) will be $ib + \frac{occ}{4b}$; ſo that a right line drawn from the *Focus* to the end of the ordinate, will always be equal to the abſciſſa EF +

† EN *i. e.* (if EO be made equal to EN) to the compounded line FO.

SCHOLIVM III.

HEnce you have an eaſier way of deſcribing the parabola in *Plano* from the given *Focus* and *Vertex*, *viz.* (*Fig:* 117.) the axis being prolonged thro' the vertex E to O, ſo that EO ſhall = EN, if a ruler HI be ſo moved by the hand G, according to PQ, from OF to HI, that putting in a ſtyle or pin, it ſhall always keep the part of the Thred NGI (which muſt be of the ſame length with the rule HI) as faſt as if it were glued to it (which perhaps might alſo be done with the Compaſſes by an artifice which we will hereafter alſo accommodate to the hyperbola) and at the ſame time it will deſcribe in *Plano* the part of the line EGR. That this will be a parabola is evident from the foregoing Conſect. becauſe as the whole thred is always = to the ruler IH; ſo the part GN is always neceſſarily equal to the part GH, *i. e.* to the line FO. Moreover from the ſame ſixth Conſect. and *Fig.* 116. may be drawn another eaſie way of deſcribing the parabola in *Plano* from the *Focus* and Vertex given thro' innumerable points G to be found after the ſame way: *viz.* If from any aſſumed point in the ax F you draw to the ax a perpendicular, and at the interval FO from the *Focus* N you make an interſection in G. Which innumerable points G will be determined with the ſame facility, having given only, or aſſumed the axis and *Latus Rectum*, by vertue of the preſent Propoſition. For if, having aſſumed at pleaſure the point F in the axis, you find a mean proportional between the *Latus Rectum* and the abſciſſa EF, a ſemiordinate FG made equal to it, will denote or mark the point G in the parabola ſought.

Propoſition II.

IN the hyperbola (GKEH Fig 118.) (a) the ſquare of the ſemiordinate (IK) is equal to the rectangle (IL) made of the Latus Rectum (EL) and the abſciſſa (EI) together with another rectangle LS of the ſaid abſciſſa (EI or LR) and RS a fourth proportional to DE the Latus Tranſverſum, (EL) the Latus Rectum, and EI the Abſciſſa.

(a) I. *Property of the Hyperb. Apoll. Prop.12.Lib.1.*

Demonſtration.

Suppoſe the ſide of the cone AB here alſo = a, and BM parallel to the ſection = b, and the intercepted line AM = c, and EI = eb; all according to the analogy we have obſerved in the parabola; and NI will be as there = ec. Making moreover MC = d and the *Latus Tranſverſum* DE = ob, ſo that DI ſhall be = $ob + eb$; then will (by reaſon of the ſimilitude of the △ △ BMC, DEP, and DIO) EP be = od, and IO = $od + ed$, and ſo QO = ed. Therefore □ NIO will be = $oecd + eecd$ = □ IK. But this ſquare divided by the Abſciſſa EI = eb gives $\frac{oecd + eecd}{eb}$ or $\frac{ocd + ecd}{b}$ for the line IS which with the abſciſſa would make the rectangle ES = to the ſaid ſquare IK. Now therefore, if here alſo we call a Line the *Latus Rectum* found after the ſame way as in the parabola, *viz* by making

as b —— to c —— ſo od to a fourth $\frac{odc}{b}$

(as a line parallel to the ſection —— to the intercepted diam. ſo the *Latus Primarium*, but that the other part $\frac{ecd}{b}$ will be a fourth proportional to bc and ed or to eb, ec and ed, or (to ſpeak with *Apollonius* as we have done in the Prop.) to ob, $\frac{ecd}{b}$ and eb (for in theſe three caſes you l have the ſame fourth $\frac{ecd}{b}$.) Wherefore now it is evident that the ſquare of the ſemiordinate $oecd + eecd$ is equal to the rectangle IL (made by the *Latus Rectum* $\frac{ocd}{b}$ into the abſciſſa $eb = oecd$) together with ▭ LS of this fourth proportional $\frac{ecd}{b}$ into the ſame abſciſſa Eb, which is $= eecd$. Which was to be found and demonſtrated.

CONSECTARYS.

I. HEnce you have in the firſt place the reaſon why *Apollonius* called this Section an Hyperbola; *viz.* becauſe the ſquare of the ordinate IK exceeds or is greater than the rectangle of the *Latus Rectum* and the Abſciſſa.

II. Since therefore the *Latus Rectum* here alſo as well as in the parabola is found by making as b to c ſo od to $\frac{odc}{b}$ (*i.e.* as the parallel to the ſection BM is to the intercepted Diam. AM ſo is the *Latus Primarium* EP to a fourth EL.) If any one had rather expreſs this *Latus Rectum* after *Apollonius*'s way, he will eaſily perceive, this quantity being found and multiplyed both Numerator and Denominator by b the parallel to the ſection, there will come out the equivalent quantity $\frac{obcd}{bb}$ which gives us inſtead of the former proportion this other,

as bb —— to cd —— ſo ob } to a fourth;
□ BM —— ▭ AMC—*Latus Tranſverſum* }

which is that of *Apollonius* in *Prop* 12. *Lib.* 1. and conſequently herein confirms our former.

III. You may alſo have this *Latus Rectum* geometrically, by finding a third proportional (as we have done in the parabola *Conſect.* 4. *Prop* 1.) to the abſciſſa EI (*Fig.* 119.) and the ſemiordinate IK ($=$ EF;) and then find a fourth proportional EL to DI (the ſum of the *Latus Tranſverſum* and abſciſſa) and FH already found, or IS equal to it, and DE (the *Latus Tranſverſum*) and that will be the *Latus Rectum* ſought.

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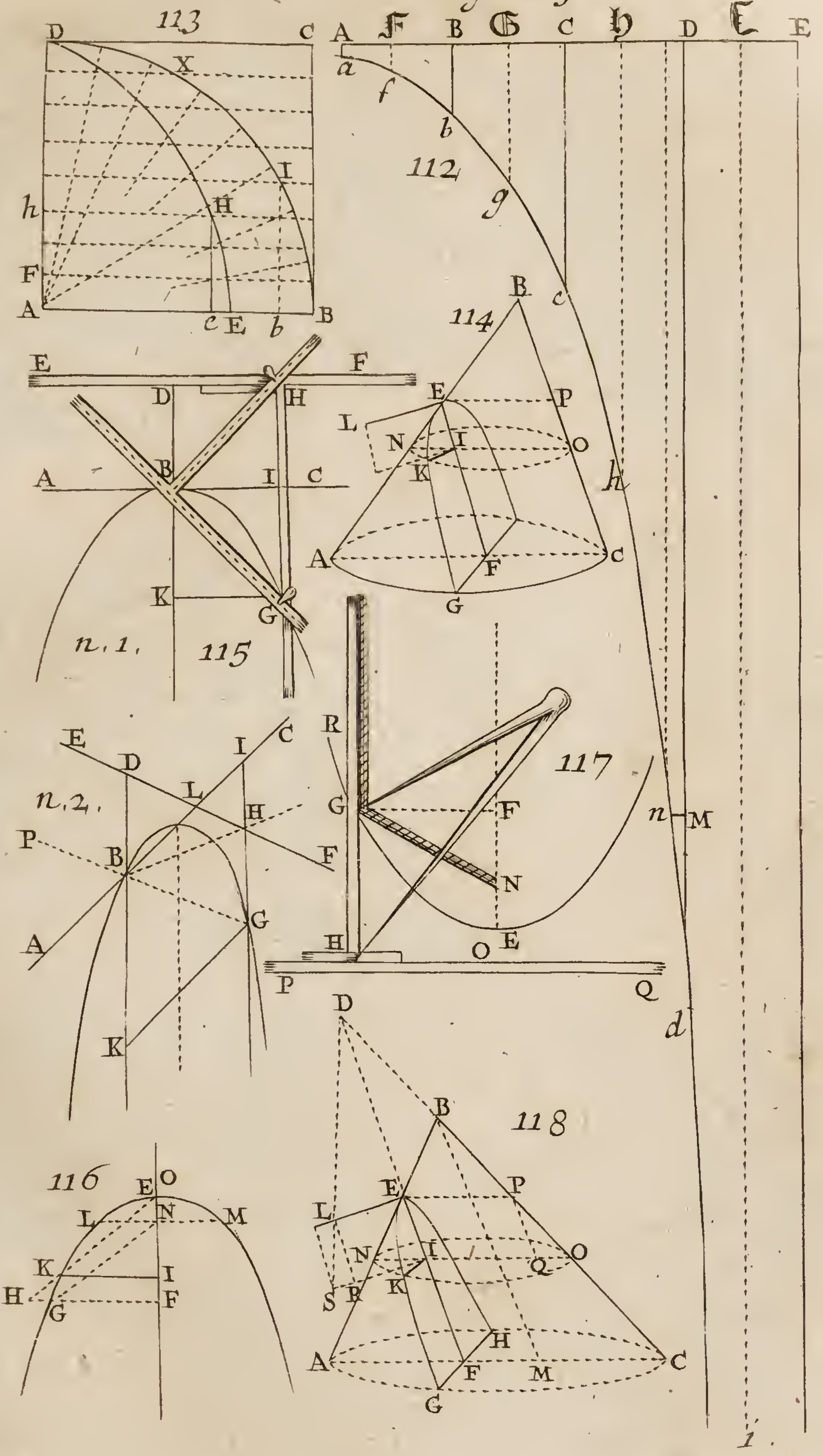
113
D C X I h H F A c E b B
112
A 𝔉 B 𝔊 C 𝔥 D 𝔈 E
a f b g c h n M d i
114
B E P L N I O K A F G C
115
E F D H A B I C K G
n. 1.
n. 2.
E D I C L H P B F G A K
117
R G F N H O E P Q
116
E O L N M K I H G F
118
D B E P L N I Q S R K H A F M G C

SHCOLIUM.

From this third Conſectary, we may reciprocally from the *Latus Rectum* and tranſverſe given, find out and apply as many ſemiordinates to the ax as you pleaſe, and ſo deſcribe the hyperbola thro' their (ends or) infinite points: *viz.* if aſſuming any part of the abſciſſa EI, you make as DE to EL ſo DI to IS; and then find a mean proportional IK between IS and the abſciſſa EI, and that will be the ſemiordinate ſought: And both this praxis and the Conſect. may be abundantly proved by ſetting it down in, and making uſe of, the literal *Calculus*.

Propoſition III.

In the Ellipſis (KDEK, Fig. 120.) the (α) ſquare of the ſemiordinate (IK) is equal to the rectangle (IL) of the Latus Rectum (EL) and the abſciſſa (EI) (leſs or) taking firſt out another rectangle (LS) of the ſame abſciſſa (EI or LR) and RS a fourth proportional to (DE) the Latus Tranſverſum (EL) the Latus Rectum and (EI) the abſciſſa.

(α) 1. *Propert. of the Ellipſe of Apoll. Prop.* 13. *Lib.* 1.

Demonſtration.

Suppoſe the ſide of the cone to be AB here alſo = *a* and BM parallel to the ſection = *b* and the intercepted AM = *c*, and EI = *eb*; and NI will be again = *ec*, all as in the hyperbola. And making alſo here as in the hyperbola MC = *d*, and the *Latus Tranſverſum* DE = *ob*, ſo that DI will be *ob* — *eb*; then will (by reaſon of the ſimilitude of the △ △ BMC, DEP and DIO) EP be = *od*, and IO = *od* — *ed*. Therefore ▭ of NIO will be = *oecd* — *eecd* = □ IK. But this ſquare divided by the abſciſſa EI = *eb* gives $\frac{oecd - eecd}{eb}$ or $\frac{ocd - ecd}{b}$ for that line IS which with the abſciſſa would make the rectangle ES = to the ſaid ſquare IK. Now therefore if we call the *Latus Rectum* a right line found after the ſame way as in the parabola, by making according to *Conſ.* 1. *Prop.* 1.

as *b* to *c* —— ſo *od* — to a fourth $\frac{ocd}{b}$ *i. e.* as the line parallel to the ſection —— to the intercepted diameter —— ſo the *Latus Primarium, &c.* It is manifeſt that the *Latus Rectum* is one part of the line juſt now found; and the other part $\frac{ecd}{b}$ is a fourth proportional to *b*, *c* and *ed*, or (to ſpeak with *Apollonius* as we have done in the Prop.) to *ob*, $\frac{ocd}{b}$ and *eb* (for there will come out the ſame quantity $\frac{ecd}{b}$;) wherefore now it is evident that the □ of the ſemiordinate IK is equal to the ▭ IL (of the *Latus Rectum* $\frac{ocd}{b}$ and the abſciſſa *eb* = *oecd)* having firſt taken out thence the ▭ LS, or *eecd* out of that fourth proportional $\frac{ecd}{b}$ by the ſame abſciſſa *eb*; which was to be found and demonſtrated.

CONSECTARYS.

I. HEnce yov have firſt of all the reaſon of the name of the *Ellipſe*, which *Apollonius* gave to this ſection; *viz.* becauſe the ſquare of the ſemiordinate IK is defective of, or leſs than the rectangle of the *Latus Rectum* and the abſciſſa.

II. Since therefore the *Latus Rectum* here alſo as well as in the parabola and hyperbola, is found by making as *b* to *c* ſo *od* to $\frac{ocd}{b}$ (*i. e.* as BM parallel to the ſection is to the intercept. diam. AM ſo the *Latus Primarium* EP to a fourth EL) now if any one had rather expreſs this *Latus Rectum* after *Apollonius*'s way, he will eaſily ſee that the quantity above found being multiplyed both Numerator and Denominator by *b*, that there will come out an equivalent one $\frac{obcd}{bb}$, which inſtead of the former proportion will give this other,

as

$$\left.\begin{array}{l}\text{as } bb \text{ —— to } cd \text{ —— ſo } ob\\ \quad BM \text{ —— } ▭\, AMC \text{ — } \textit{Latus Tranſverſ.}\end{array}\right\}\text{to a fourth;}$$

which is the ſame with that we have alſo found in the hyperbola, and which alſo *Apollonius* has *Prop.* 13. *Lib.* 1.

III. This *Latus Rectum* may alſo be had geometrically, if you find, 1. in the hyperbola a third proportional FH to the abſciſſa EI (*Fig.* 121.) and ſemiordinate IK (= EF.) 2. But EL a fourth proportional to DI (the difference of the *Latus Tranſverſum* and the abſciſſa) and the found FH, or IS equal to it, and the *Latus Tranſverſum* DE, is the *Latus Rectum* ſought.

SCHOLIUM.

FRom this third Conſect. we may reciprocally, having the *Latus Rectum* and *Tranſverſum* given, apply as many ſemiordinates to the ax as you pleaſe, and ſo draw the ellipſis thro' as many points given as you pleaſe, *viz.* if, taking any abſciſſa EI, you make as DE to EL ſo DI to a fourth IS; then between this IS and the abſciſſa EI find a mean proportional IK, and that will be the ſemiordinate ſought: And this *Praxis* alſo and the third Conſect. may be abundantly proved by making uſe of a literal *Calculus*. For *e. g.* here a fourth proportional to ob, $\frac{ocd}{b}$ and $ob - eb$ will be $\frac{ocd - ecd}{b}$; and a mean proportional between this fourth and eb will be $\sqrt{oecd - eecd}$, &c.

Propoſition IV.

IN a Parabola (α) *the ſquares of the ordinates are to one another as the abſciſſa's.*

(α) 2. *Property of the Parab.* 20. *Prop. Apoll. Lib.* 1. *Con.*

Demon-

Demonſtration.

For if EF (*Fig.* 122.) be called ib, as above EI was called eb, ſince the *Latus Rectum* is $\frac{occ}{b}$; the ſquare of FG will be $= oicc$. Therefore

$\square$ IK will be to $\square$ FG as $\left.\right\}$ e to i or
$oecc$ ———— $oicc$ $\left.\right\}$ eb to ib. Q. E. D.

CONSECTARY.

HEence having drawn LO parallel to the ax or diameter EF, if it be cut by the tranſverſe line EG in M and by the curve of the parabola in K; then will OL, ML, and KL be continual proportionals. For EF is to EN as FG to NM or IK, by reaſon of the ſimilitude of the △ △ EFG and ENM. But the ſquares FG and IK are in duplicate proportion of EF to EN by *Prop.* 35. *Lib.* 1. and are alſo in the ſame proportion as the abſciſſa's EF and EI by the preſ. Therefore EF to EI is alſo in duplicate proportion of EF to EN *i. e.*

$$\left\{\begin{matrix}EF\\OL\end{matrix}\right. \text{ is to } \left.\begin{matrix}EN\\ML\end{matrix}\right\} \text{ as } \left.\begin{matrix}EN\\ML\end{matrix}\right\} \text{ to } \left.\begin{matrix}EI\\KL\end{matrix}\right\} \text{Q. E. D.}$$

Propoſition V.

IN the hyperbola and Ellipſis (α) the ſquares of the Ordinates are as the rectangles contained under the lines which are intercepted between them, and the Vertex's of the Latus Tranſverſum's.

(α) *Property of the Hyperbola and Ellipſ. Apol.* 21. *Lib.* 1.

Demonſtration.

For, if EF (*Fig.* 118. and 120.) be called ib, as EI was above called eb, then will according to *Prop.* 2. and the 3*d.* deduction.

GF $= oicd + eicd$ in the Hyperb.
$oicd - eicd$ in the Ellipſ.
and the ▭ DFE $= oibb + iibb$ in the Hyperb.
$oibb - iibb$ in the Ellipſ.

Therefore the □ KI is to the ſquare GF as $oecd \,\underline{\dagger}\, eecd$ to $oicd \,\underline{\dagger}\, iicd$ i. e. as $oe \,\underline{\dagger}\, ee$ to $oi \dagger ii$.

and ▭ DIE is to the ▭ DFE, as $oebb \,\underline{\dagger}\, eebb$ to $oibb \,\underline{\dagger}\, iibb$ *i. e.* in like manner as

$oe \,\underline{\dagger}\, ee$ to $oi \,\underline{\dagger}\, ii$. Q. E. D.

CONSECTARY I.

IN the Ellipſis this may be more commodiouſly expreſſed apart thus; the ſquares of the ordinates (KI and GF) are as the rectangles contained under the ſegments of the Diameter (*viz.* DIE and DFE) in which ſenſe this property is alſo common to the circle, as in which the ſquares of the ordinates are always equal to the rectangles of the ſegments.

CONSECTARY II.

THerefore, if the *Latus Rectum* be conceived to be applyed in the hyperbola, ſo that N ſhall be the *Focus*; (ſee *Fig.* 123.) then will LN ⧦ $\frac{ocd}{2b}$, and its ſquare be $\frac{ooccdd}{4bb}$. But as the □ KI to the ſquare LN, ſo is the ▭ DIE to the ▭ DNE i. e. $oecd \dagger eecd$ to $\frac{ooccdd}{4bb}$ ſo is $oebb \dagger eebb$ to $\frac{oocd}{4}$. But now the ▭ of the whole DE and the part added EN into the part added EN, *i. e.* ▭ DNE (⧦ $\frac{oocd}{4}$ together with the ſquare of half CE (⧦ $\frac{oobb}{4}$) is ⧦ □ compounded of half and the part added CN ⧦ $\frac{oocd \dagger oobb}{4}$ by *Prop.* 9. *lib.* 1. Wherefore CN the diſtance of the *Focus* from the centre is ⧦ $\sqrt{\frac{oodc \dagger oobb}{4}}$. But $\frac{oocd}{4}$ is the fourth part of the ▭ of the *Latus Tranſverſum* ob and the *Latus Rectum* $\frac{ocd}{b}$ (or the fourth part of the figure, as *Apollonius* calls it) and $\frac{oobb}{4}$ is the □ of $\frac{ob}{2}$ i. e. of half the *Latus Tranſverſum.* Wherefore we have found the following Rule of determining the *Focus* in an hyperbola: *If a fourth part of the figure* (or the rectangle of the *Latus Rectum* into the *Tranſverſum) be added to the ſquare of half the* Latus Tranſverſum, *and from the ſum you extract the ſquare root; that will be the diſtance of the Focus from the center CN: And hence ſubſtracting half the* Latus Tranſverſum CE, *you will have diſtance of the Focus from the Vertex EN.*

CONSECTARY III.

IN like manner in the Ellipſis having drawn the ordinate LM (*Fig.* 124.) that the *Focus* may be in N, the □ LN would be $\frac{ooccdd}{4bb}$ as above, and by a like inference ▭ DNE $=$ $\frac{oocd}{4}$. But now ▭ DNE together with the ſquare of the difference CN is equal to the □ of half CE by *Propoſ.* 8. *lib.* 1. and conſequently the □ CN is $=$ □ CE — ▭ DNE, that is, $\frac{oobb}{4} - \frac{oocd}{4}$. Wherefore CN the diſtance of the *Focus* from the centre is $= \sqrt{\frac{oobb - oocd}{4}}$. Wherefore we have found the following Rule to determine the *Focus* in the Ellipſe. *If the fourth part of the figure* (or the rectangle of the *Latus Rectum* into the *Latus Tranſverſum) be ſubſtracted from the ſquare of half the* Latus Tranſverſum, *and from the remainder you ſubſtract the ſquare root; that will be the diſtance of the Focus from the Centre CN: And taking hence half the* Latus Tranſverſum CE, *you'l have the diſtance of the Focus from the Vertex EN.*

SCHOLIUM I.

BOth the Rules are eaſie in the practice, for ſince $\frac{oobb}{4}$ is nothing but the ſquare of CE, and $\frac{oocd}{4}$ nothing but the rectangle of $\frac{1}{4}$ DE into LM; if between LM and $\frac{1}{4}$ DE or MO (*Fig.* 125.) you find a mean proportional MN, (and ſo whoſe □ is equal to that ▭) and in the hyperbola join to it at right angles MC = CE, the hypothenuſa CN will be the diſtance ſought of the *Focus* from the centre: And the ſame may be had in the Ellipſis, if (*n.* 2.) having deſcribed a ſemi-circle upon CM = CE you draw or apply the mean found MN, and draw CN.

SCHOLIUM II.

HEnce alſo we have (α) a *new geneſis of the Ellipſe in Plano* about the diameters given, from the ſpeculations of Monſieur *de Witt*; *viz.* If about the rectilinear angle DCB (*Fig.* 126. *n.* 1 and 2.) conſider'd as immoveable, the rule NLK (which all of it will equal the greater ſemidiameter CB, and with the prominent part LK the leſſer CD) be ſo moved that N going from C to D, and L from B to C may perpetually glide along the ſides of the angle, the extreme point in the K in the mean while deſcribing the curve BKE (and in a like application the other quadrants) and that this curve thus deſcribed will be the ellipſis of the ancients is hence manifeſt, becauſe it has the ſecond property of the ellipſe juſt now deſcribed. For, 1. if the angle DCB or NCB be ſuppoſed to be a right one (as in *Fig.* 126. *num.* 1.) and the rule KN in the ſame ſtation, it marked out the point K, and having apply'd the ſemiordinate KI, and drawn the perpendicular LM, from the ſquare KL and the ſquare CE (as being equal) ſubſtract mentally the equal ſquares LM and CI, and there will remain by virtue of the Pythagorick Theor. on the one hand □KM and on the other by

(α) *De Witt Elem. Curv. Lib.* 1. *Cap.* 3. *Prop.* 13.

Prop 8. *lib.* 1. ▭ DIE equal among themſelves. But now the ſquare of KI is to the ſquare of KM (*i e.* to the ▭ DIE) as the ſquare of KN to the ſquare of KL (*i. e.* as the ſquare of CB to the ſquare of CE) by reaſon of the ſimilitude of the △ △ KLM and KNI; and ſince the ſame may be demonſtrated after the ſame manner of any other ſemiordinate K*i* *viz.* that its □ K*i* is to the ▭ D*i*E as the ſquare CB to the ſquare CE. It alſo follows, that the □ KI is to the ▭ DIE as □ K*i* to the ▭ D*i*E, and alternatively, the ſquare KI will be to the ſquare K*i* as the ▭ DIE to the ▭ D*i*E; which is the ſecond property of the ellipſe. 2. If the angle NCE be not a right one (as in *Fig.* 126.*n.*2. and the like caſes) having drawn NO and KP parallel to the rule *nl*B in the firſt ſtation, [in which ſtation the angle NCE, to which the flexible ruler is to be made, is determined, *viz.* by letting fall the perpendicular B*l* from the extremity of one diameter upon the other, and moreover by adding or ſubſtracting the difference of the ſemi-diameters *ln*] having alſo drawn the Ordinate KIM, and PI parallel to CN; which being done the △ △ CB*l* and IKF, and alſo CB*n* and IKP will be ſimilar. Wherefore having joined NP, from the paralleliſm of the lines IP and NC and the ſimilitude of the aforeſoid △ △, as alſo of NCO and *n*C*l*, it will be eaſie to conclude that NCIP is a parallelogram. Wherefore ſince KN is = CE and □ KN = □ CE, having ſubſtracted the ſquares of the equal lines NP and CI, there will remain on the one hand □ KP on the other the ▭ DIE equal among themſelves as above. Therefore the ſquare of KI will be to the ſquare of KP (*i. e.* to the ▭ DIE) as the ſquare of BC to the ſquare of B*n* (*i. e.* to the ſquare of CE) as in the former caſe: And ſince here alſo the ſame may be demonſtrated after the ſame manner of any other ſemiordinate K*i*; we may infer as above, that the □ □ KI and K*i* are to one another as the rectangles DIE and D*i*E, *&c.*

But after what way the ſame ellipſes may be deſcribed by theſe right lined angles without any of theſe rulers thro' infinite points given, will be be manifeſt from the ſame figures to any attentive Perſon. For having once determined the angle NCE or *n*CD (*num.* 2. *e. g.*) if NL or *nl* be applyed where you pleaſe by help of a pair of compaſſes, and continued to

K,

K, ſo that LK or *lk* ſhall be equal to *l*B, you will have every where the point K, *&c.*

CONSECTARY IV.

Since in the hyperbola (*Fig* 123.) the $\square CN - \square CE = \square DNE$, and in the ellipſis (*Fig* 124.) $\square CE - \square CN = \square DNE$, by vertue of *Prop.* 9 and 8. *lib.* 1 if for CN you put on both ſides for brevity's ſake *m*, then will the $\square$ DNE in the hyperbola be rightly expreſſed in theſe terms $mm - \frac{oobb}{4}$, and in the ellipſis in theſe $\frac{oobb}{4} - mm$.

Propoſition VI.

In the parabola (α) *the Latus Rectum is to the ſum of two ſemiordinates (e. g.* IK + FG i. e. *HO in Fig.* 122.) *as their difference (OG) to the difference of the abſciſſa's (IF or KO.)*

(α) 3. *Property of the Parab.*

Demonſtration.

For if the greater abſciſſa EF be made $= ib$, and the leſs EI $= eb$, the ſemiordinates anſwering to them FG and IK will be $\sqrt{ooic}$ and $\sqrt{oecc}$ as is deduc'd from *Prop.* 1. Wherefore if you ſet in the ſame ſeries

1	2	3	4
The *Latus R.*	— ſum of the ſemiord.	— their diff.	— diff. of the abſciſſ.
$\frac{occ}{b}$	$- \sqrt{oicc} + \sqrt{oecc}$	$- \sqrt{oicc} - \sqrt{oecc}$	$- ib - eb$

and multiply the extremes and means, you'l have on both ſides the ſame product $oicc - oecc$, which will prove by vertue of *Prop.* 19. *lib.* 1. the proportionality of the ſaid quantities. Q. E. D.

SCHOLIUM.

THis is that property of the parabola, whereon the *Clavis Geometrica Catholica* of Mr. *Thomas Baker* is founded, which as unknown to the ancients, nor yet taken notice of by *Des Cartes*, he thinks was the reaſon why that incomparable Wit could not hit upon thoſe univerſal rules for ſolving all Equations howſoever affected. Concerning which we ſhall ſpeak further in its place. We will only further here note, that *Baker* was not the firſt Inventor of this property, but had it, as he himſelf ingeniouſly confeſſes, out of a Manuſcript communicated to him by *Tho. Strode* of *Maperton*, Eſquire.

Propoſition VII.

IN the hyperbola and ellipſis (α) *the Latus Rectum is to the Latus Tranſverſum, as the ſquare of any ſemiordinate (e. g. IK in Fig.* 118 *and* 120*) to the rectangle (DIE) contained under the lines intercepted between it and the Vertex's of the Latus Tranſverſum.*

(α) *The* 3. *Property of the Hyperb. and Ellipſ. Apollon. lib.* 1. *Prop.* 21. *pars prior.*

Demonſtration.

For the *Latus Rectum* is on both ſides $\frac{ocd}{b}$, the *Latus Tranſverſum ob*, &c. Wherefore if you make in the ſame ſeries as the *Latus R.* to the *Lat. Tranſv.* ſo the □ IK to ▭ DIE

$\frac{ocd}{b}$ — *ob* —— in hyperb. *oecd* † *cecd* —— *oebb* † *eebb*

in ellipſ. *oecd* — *eecd* —— *oebb* — *eebb*

The rectangles of the extremes and means will both be *ooebcd* † *oeebcd*, and ſo will prove the proportionality of the ſaid quantities, by *Prop.* 19. *lib.* 1. Q. E. D.

CONSECTARY I.

HEnce having given in the ellipſis (ſee *Fig.* 124) the *Latus Rectum* and the tranſverſe ax, you may eaſily obtain the ſecond ax or diameter, if you make
as the *Lat. Tranſv.* to the *Lat. Rect.* ſo the □ DCE to □ AC

$$ob \;—\; \frac{ocd}{b} \;—\; \frac{oobb}{4}\ \text{F.} \;—\; \frac{oocd}{4}.$$

CONSECTARY II.

THerefore the □ of the whole AB will be $=$ *oocd* $=$ □ of the *Latus Rectum* into the *Lat. Tranſv.* (which *Apllonius* calls the *Figure*) ſo that the ſecond Ax (and any ſecond Diameter) will be a mean proportional between the *Latus Rectum* and the *Latus Tranſverſum.* Hence in the hyperbola alſo the ſecond or conjugate diameter may be called a mean proportional between the *Latus Rectum* and *Tranſverſum*, i. e. $\sqrt{oocd}$ or a line which is equal in power to the *Figure*, as *Apollonius* ſpeaks.

SCHOLIUM I.

HEnce may be derived another and more ſimple way of delineating organically the ellipſis in *Plano* about the given axes AB, DE (*Fig* 127.) which *Schooten* has given us; *viz.* by the help of two equal rulers CG and GK moveable about the points G and G: If, *viz.* the portions CF and HK are equal to half the leſſer ax AC, but taken with both the augments (*viz.* CF † FG † GH) may $=$ $\frac{1}{2}$ the greater ax CD or CI; and the point K moving along the produced line DE the point H may deſcribe the curve EHAD. That this will be an ellipſis will be evident by vertue of this ſeventh *Prop.* from a property that agrees to this curve in all its points H. For having drawn circles about each diameter, and the lines IHN, FO perpendicular to CE; and having made the *Latus Rectum* EL, which is a third proportional to DE and AB by the ſecond Conſect. of this Prop. *&c.* by reaſon of the ſimilitude of the triangles CFO, CIN, FO will be to FC as

IN to IC, and alternatively FO to IN as FC to IC *i. e.* as AC to CE or AB to DE. Therefore alſo the ſquare of FO(or HN) will be to the ſquare of IN, as the ſquare of AB to the ſquare of DE, by *Prop.* 22. *lib.* 1. *i. e.* as EL the *Latus Rectum* to ED the *Latus Tranſverſum*, by *Prop.* 35. *lib.* 1. But the □ IN is $=$ DNE from the proporty of the circle. Therefore □ FO (or of the ſemiordinate HN) is to the ▭ DNE as EL the *Latus Rectum* to ED the *Latus Tranſv.* therefore by vertue of the preſ. *Prop.* the point H is in the *Ellipſis*, and ſo any other, *&c.* Q. E. D.

CONSECTARY III.

NOW if in the ellipſis the □ of AC the ſecond Ax ($= \frac{oocd}{4}$ by *Conſect.* 1.) and □ CN the diſtance of the *Focus* from the centre ($= \frac{oobb - oocd}{4}$ by *Cenſect* 3. *Prop.* 5. the figure whereof you may ſee *n.* 124.) be joined in one ſum; the □ AN will be $= \frac{oobb}{4}$, and ſo the line AN $= \frac{ob}{2}$ i. e. to half the *Latus Tranſverſum*: So that hence having the axes given you may find the *Foci*, if from A at the interval CD you cut the tranſverſe ax in N and N.

CONSECTARY IV.

NOW if, on the contrary, in an hyperbola (*Fig.* 123.) the □ AC or EF $= \frac{oocd}{4}$ be ſubſtracted from the □ CF or CN $= \frac{oobb + oocd}{4}$ by vertue of *Conſect.* 2. *Prop.* 5. there will remain $\frac{oobb}{4}$ and its root $\frac{ob}{2}$, *i. e.* half the *Latus Tranſverſum* CD: So that here alſo, the axes being given, you may find the *Focus*'s, if from the vertex E you make EF a perpendicular to the ax $=$ to the ſecond Ax AC, and at the interval CF from the centre C you cut the *Latus Tranſverſum* continued in N and N.

SCHOLIUM II.

BUT now, that the right lines KN and KN drawn from any other point (*e. g.* K) to the *Foci*, when taken together in the ellipſis, but when ſubſtracted the one from the other in the hyperbola, are equal to the *Latus Tranſverſum* DE, we will a little after demonſtrate more univerſally, and alſo ſhew an eaſie and plain *Praxis* of delineating the ellipſis and hyperbola in *Plano*, having the axes and conſequently the *Foci* given.

CONSECTARY V.

SInce we have before demonſtrated *Conſect.* 2. and 3. *Prop.* 5. that the ▭ DNE in the hyperbola and alſo in the ellipſis is $\asymp \frac{oocd}{4}$; and here in *Conſect.* 1. the □ of the ſecond ſemi-diameter AC is alſo $\asymp \frac{oocd}{4}$; it is evident that this □ AC is equal to the ▭ DNE.

CONSECTARY VI.

IT is hence moreover evident, if the ſquare of half the tranſverſe diameter GE $\asymp \frac{oobb}{4}$ be compared with the ſquare of half the ſecond diameter AC or EF $\asymp \frac{oocd}{4}$, multiplying both ſides by 4 and dividing by o; they will be to one another as obb to ocd i. e. further dividing both ſides by b, as ob to $\frac{ocd}{b}$ the *Latus Tranſverſum* to the *Latus Rectum*.

CONSECTARY VII.

BUT ſince alſo the ▭ DIE is to the □ IK as the *Latus Tranſverſum* to the *Latus Rectum*, by vertue of the preſent 7. *Prop.* the ſquare of CE the tranſverſe ſemidiam. will be to the ſquare of AC the ſecond ſemidiam. (or by the *5th. Conſect.* of this, to the □ DNE) as the □ DIE to the ſquare of IK.

CONSECTARY VIII.

YOU may alſo now have the □ IK (which otherwiſe in the hyperb. is $eocd \dagger eecd$, in the ellipſe $oecd - eecd$, by vertue of *Prop.* 2. and 3.) in other terms, if you make as □ CE to the □ DNE ſo the □ DIE to a fourth; *i. e.*

as $\frac{oobb}{4}$ to $mm - \frac{oobb}{4}$ } by vertue of *Conſect.* 4. *Prop.* 5.
(ſo $oebb \dagger eebb$ in the hyperb.
as $\frac{oobb}{4}$ to $\frac{oobb}{4} - mm$ ſo $oebb - eebb$ in the ellipſis.

For hence by the Golden Rule the ſquare IK may be infer'd as a fourth proportional.

In the hyperbola $\frac{4emm}{o} \dagger \frac{4eemm}{oo} - oebb - eebb$;

In the ellipſis $oebb - eebb - \frac{4emm}{o} \dagger \frac{4eemm}{oo}$:

The uſe of which quantities will preſently appear.

Propoſition VIII.

THE $\left\{\begin{array}{l}\textit{Aggregate in the ellipſe}\\ \textit{Difference in the hyperb.}\end{array}\right\}$ *of the right lines (a) KN and Kn (Fig. 128.) drawn from the ſame point K to both the Focus's is equal to the tranſverſe ax DE.*

(a) *Apollon. Lib.* 3. *Prop.* 51. *and* 52.

An Ocular Demonſtration.

WHich conſiſts wholly in this to find the lines KN and K*n* by help of the right-angled triangles IKN and IK*n* (ſc. the hypothenuſes having the ſides given) and afterwards ſee if the ſum of both in the ellipſe, and difference in the hyperbola be = *ob*, *i. e.* to the tranſverſe ax DE.

I. *In*

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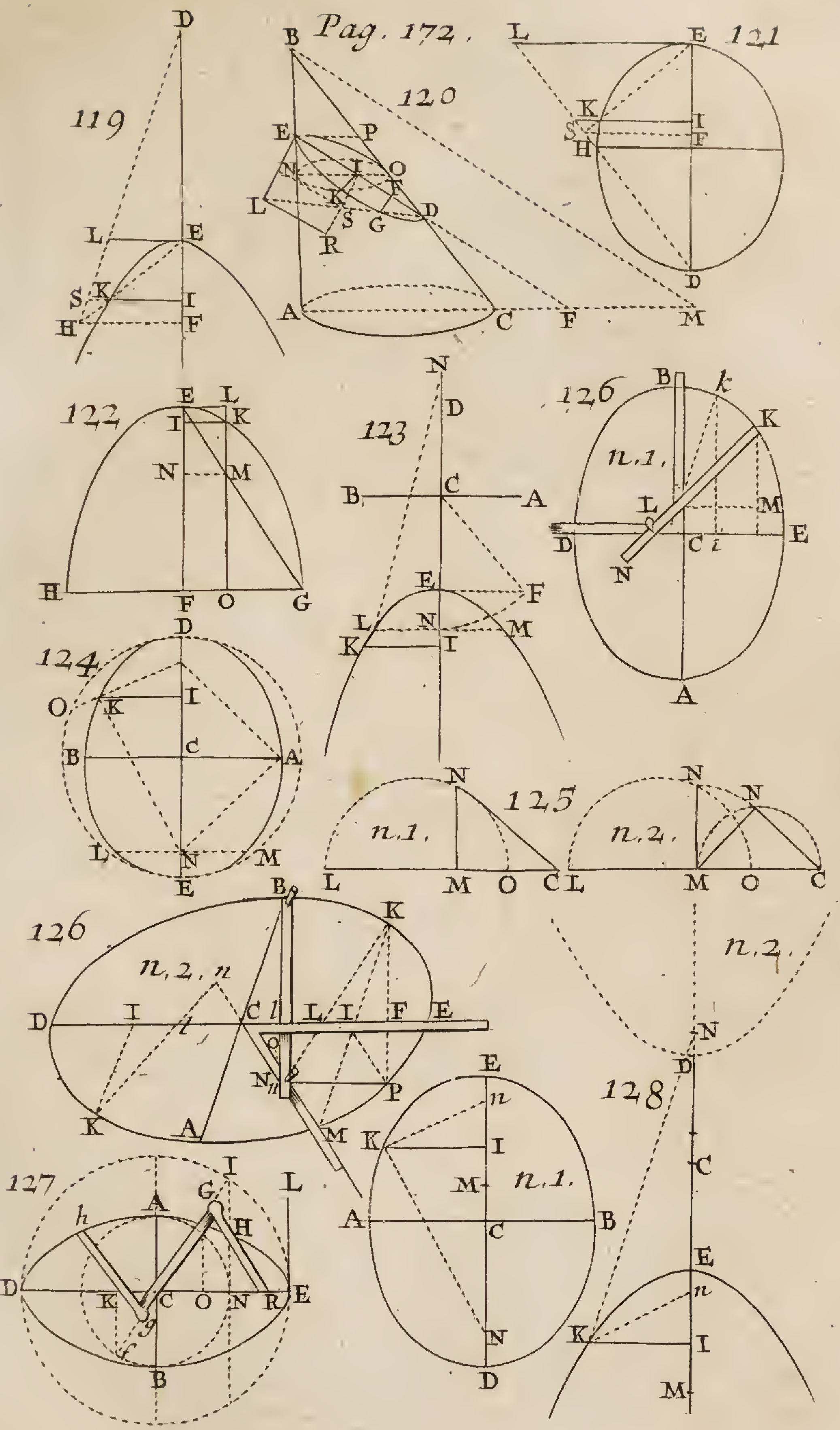
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I. *In the Ellipſis.*

Putting for CN (which above *Prop.* 5. *Conſ.* 3. was found to be $\sqrt{\frac{oobb - oocd}{4}}$) I ſay putting for it *m*, you'l have

$$IN = CI \dagger CN = \tfrac{1}{2}ob - eb \dagger m$$
$$In = Cn - CI = m - \tfrac{1}{2}ob \dagger eb$$

Therefore $\square\, IN = \tfrac{1}{4}oobb - oebb \dagger eebb \dagger obm - 2ebm \dagger mm$
$\square\, In = \tfrac{1}{4}oobb - oebb \dagger eebb - obm \dagger 2ebm \dagger mm$

Add to each $\square$ IK which was found in *Prop.* 7. *Conſect.* 8. in the ellipſis $= oebb - eebb - \frac{4emm}{o} \dagger \frac{4eemm}{oo}$ and you'l have

$$\square\, KN = \tfrac{1}{4}oobb \dagger obm - 2ebm \dagger mm - \frac{4emm}{o} \dagger \frac{4eemm}{oo}$$

and by extracting the roots (which is eaſie) you'l have

$$KN = \tfrac{1}{2}ob \dagger m - \frac{2em}{o} \text{ and}$$
$$Kn = \tfrac{1}{2}ob - m \dagger \frac{2em}{o};$$

Sum *ob*. Q.E.D.

II. *In the Hyperbola.*

Putting again *m* for CN (which above *Conſ.* 2. *Prop.* 5. was found to be $\sqrt{\frac{oodc \dagger oobb}{4}}$) and you'l have

$$IN = CI \dagger CN = \tfrac{1}{2}ob \dagger eb \dagger m$$
$$In = CI - Cn = \tfrac{1}{2}ob \dagger eb - m. \text{ Therefore}$$
$$\square\, IN = \tfrac{1}{4}oobb \dagger oebb \dagger eebb \dagger obm \dagger 2ebm \dagger mm \text{ and}$$
$$\square\, In = \tfrac{1}{4}oobb \dagger oebb \dagger eebb - obm - 2ebm \dagger mm$$

Add to both the $\square$ IK which was found in *Prop.* 7. *Conſect.* 8. in the hyperbola $\frac{4emm}{o} \dagger \frac{4eemm}{oo} - oebb - eebb$ and you'l have

$$\square\, KN = \tfrac{1}{4}oobb \dagger obm \dagger 2ebm \dagger mm \dagger \frac{4emm}{o} \dagger \frac{4eemm}{oo}$$

Q $\square$ Kn

$$\square\, Kn = \tfrac{1}{4}oobb - obm - 2ebm + mm + \frac{4emm}{o} + \frac{4eemm}{oo}$$

and extracting the roots out of theſe (which is eaſie) you'l have $KN = \frac{1}{2}ob + m + \frac{2em}{o}$

$$Kn = \tfrac{1}{2}ob - m - \frac{2em}{o}$$

(which is a falſe or impoſſible root, for it would be CE—CN and moreover —— another quantity.

Or $Kn = m + \frac{2em}{o} - \frac{1}{2}ob$; which is a true and poſſible root.

The difference therefore of the true roots is $= ob$. Q. E. D.

SCHOLIUM I.

WE firſt of all tried to make a literal Demonſtration by uſing the quantity of the ſquare IK as you have it expreſſed *Prop.* 2. and 3. and the quantity IN as it was compounded of $CI = ob + eb + CN = \sqrt{\frac{oocd + oobb}{4}}$, *&c.* but we found it very tedious in making only the ſquares of IN and In. Then for the ſurd quantity CN we ſubſtituted another, *viz.* m, and we produced the ſquares of IN and In as above, but we added the ſquare of IK in its firſt value; and thus we obtain'd the ſquares KN and Kn, but in ſuch terms, that the exact roots could not be extracted, but muſt be exhibited as ſurd quantities, and conſequently we muſt make uſe of the rules belonging to them to find their ſum or difference, which we laid down *Conſ.* 3. *Prop.* 7. and *Conſect. Prop.* 10. *Lib.* 1. which tho' it would ſucceed, yet would be full of trouble and tedioúſneſs. Therefore at length when we came to uſe thoſe other terms which expreſs the ſquare IK, the buſineſs ſucceeded as eaſie as we could wiſh, and that in a plain and eaſie way and no leſs pleaſant, which I doubt not but will alſo be the opinion of the Reader, who ſhall compare this with other demonſtrations of the ſame thing, which only lead indirectly to this truth, or with them which *de Witte* has given us in *Elem. Curvar. lin. p. m.* 293. and 302. and which he thinks eaſie and ſhort enough in reſpect of others both of the ancients and moderns, and which we have reduced into this yet more diſtinct form, and accommodated to our ſchemes.

Preparation for the Hyperbola.

Make as $\left\{\begin{matrix}\text{CD to CN}\\ \text{EE — CN}\end{matrix}\right\}$ ſo CI to CM

ſo that the ▭ of $\left\{\begin{matrix}\text{DCM}\\ \text{MCE}\end{matrix}\right\}$ will be = ▭ $\left\{\begin{matrix}\text{NCI}\\ n\text{CI.}\end{matrix}\right.$

Becauſe therefore it will be by *Conſect.* 7. *Prop.* 7.

as □ CD to ▭ DNE, ſo the ▭ DIE to the □ IK.

And alſo by compoſition,

as □ CD to $\left\{\begin{matrix}\square\text{ CD † }\square\text{ DNE}\\ \textit{i. e. }\square\text{ CN }\textit{per}\text{ 9 }\textit{lib}\text{ 1.}\end{matrix}\right\}$ ſo DIE to DIE † □ IK.

Therefore by a Syllepſis,

as □CD to the □CN ſo $\left\{\begin{matrix}\square\text{ CD † DIE}\\ \textit{i. e. }\square\text{ CI}\end{matrix}\right\}$ to □ CN † DIE † □ IK.

But alſo by the Hypotheſis.

as the □ CD to the □ CN ſo □ CI to □ CM. Therefore □ CM is = □ CN † DIE † □ IK.

Demonſtration.

Since therefore it is certain that the difference between DM and EM is the tranſverſe ax DE; if it be demonſtrated that DM is = KN and EM = K*n*, the buſineſs will be done, becauſe the difference between KN and K*n* is alſo the tranſverſe ax DE.

Reſolve the □ KN.

It is certain that NI*q* † IK*q* = KN*q*.

Subſtitute for NI*q*, by the 7. *Lib.* 1. CI*q* † CN*q* † 2NCI.

Preparation for the Ellipſis.

Make as CD to CN ſo CI to CM.

So that the ▭ DCM is = ▭ NCI.

Becauſe therefore by *Conſect.* 7. *Prop.* 7.

as □ CD to ▭ DNE ſo ▭ DIE to the □ IK;

Then alſo by dividing,

$$\text{as } \square\,CD \text{ to} \left\{\begin{array}{l} \square\,CD - DNE \\ \textit{i. e. } \square\,CN,\text{ by 8. } \textit{l.}\text{ 1.} \end{array}\right\} \text{ſo } DIE \text{ to } DIE - \square\,IK.$$

Therefore by a Dialepſis,

$$\text{as } \square\,CD \text{ to } \left\{\begin{array}{l} \square\,CD - \square\,DIE \\ \textit{i. e. } CI\,\square \text{ by 8. } \textit{cit.} \end{array}\right\} \text{to } \square\,CN - DIE \dagger \square\,IK,\ \square\,CN, \text{ ſo}$$

But alſo by the Hypotheſis,

as □ CD to □ CN, ſo □ CI to □ CM:
Therefore □ CM is = □ CN —— DIE † □ IK.

Demonſtration.

Since therefore it is certain that the ſum of DM and EM is the tranſverſe ax DE; if it be demonſtrated that DM is = KN and EM = K*n*, the buſineſs will be done, becauſe the ſum of KN and K*n* is alſo equal to the tranſverſe ax DE.

Reſolve the □ *KN.*

It is certain that NI*q* † IK*q* = KN*q*.
Subſtitute for NI*q*, by the 7. *lib.* 1. CI*q* † CN*q* † 2NCI.
Then will CI*q* † CN*q* † 2NCI † IK*q* = KN*q*.
Subſtitute for CI*q*, by the 9. *lib.* 1. CD*q* † DIE; then will CD*q* † DIE † CN*q* † 2NCI † IK*q* = KN*q*.

Reſolve alſo □ *DM.*

$$\text{It is certain that } CMq \dagger CDq \dagger \left\{\begin{array}{l} 2DCM \\ 2NCI \end{array}\right\} = DM, \text{ by the 7 } \textit{lib.}\text{ 1.}$$

Subſtitute for CM*q* its value by the Preparation, and you'l have
CN*q* —— DIE † IK*q* † CD*q* † 2NCI = DM*q*:
Which were before = KN*q*.
Therefore KN = DM; which is one.

In like manner reſolve □ *Kn.*

It is certain that *n*I*q* † IK*q* = K*nq*.
Subſtitute for *n*I*q*, by *Conſect.* 1. *Prop.* 10. *Lib.* 1. CI*q* † CN*q* —— 2*n*CI, and you'l have
CI*q* † CN*q* —— 2*n*CI † IK*q* = K*nq*.
Subſtitute for CI*q*, by the 9. *lib.* 1. CD*q* † DIE, and you'l have CD*q* † DIE † CN*q* —— 2*n*CI † IK*q* = K*nq*.

Reſolve alſo the □ *EM.*

It is certain that 2CD*q* † 2CM*q*—DM*q* = EM*q* *per* 13. *lib.* 1.

Then will CI*q* † CN*q* † 2NCI † IK*q* = KN*q*.

Subſtitute for CI*q* by the 8. *lib.* 1. CD*q*—DIE; then will
CD*q*—DIE † CN*q* † 2NCI † IK*q* = KN*q*.

Reſolve alſo the □ *DM.*

It is certain that $CMq \dagger CDq \dagger \left\{\begin{matrix}2DCM\\2NCI\end{matrix}\right\} = DMq$ *per* 7. *lib.* 1.

Subſtitute for CM*q* its value from the preparation, and you'l have
CN*q*——DIE † IK*q* † CD*q* † 2NCI = DM*q*:
Which before were = KN*q*.
Therefore KN = DM; which is one.

In like manner reſolve the □ *Kn.*

It is certain that *n*I*q* † IK*q* = K*nq*.

Subſtitute for *n*I*q* by *Conſect.* 1. *Prop.* 10. *lib.* 1.
CI*q* † CN*q*——2NCI, and you'l have
CI*q* † CN*q*——2NCI † IK*q* = K*nq*.

Subſtitute for CI*q* *per* 8. *lib.* 1. CD*q*—DIE, and you'l have
CD*q*——DIE † CN*q*—2NCI † IK*q* = K*nq*.

Reſolve alſo □ *EM.*

It is certain that 2CD*q* † 2CM*q*——DM*q* = EM*q* *per* 13. 1.

Subſtitute the value of DM*q* firſt found above, and you'l have
CD*q* † CM*q*——2*n*CI = EM*q*.

Subſtitute for CM*q* the value as in the preparation, and you'l have
CD*q* † CN*q* † DIE——2*n*CI † IK*q* = EM*q*:
Which were before = K*nq*.
Therefore K*n* = EM; which is the other.

S C H O-

SHCOLIUM II.

HEnce you have the common mechanical ways of deſcribing the ellipſis and hyperbola about their given axes; and the ellipſis, if the *Foci* N, N, (*Fig.* 129. *n.* 1.) are given, or found according to *Conſect.* 3. *Prop.* 7. and having therein ſtuck or fixed two pins, put over them a thread NF*n* tyed both ends together preciſely of the length you deſign the greater ax DE to be of, and having put your pencil or pen in that △-ſtring draw it round, always keeping it equally extended or tight. Now becauſe the parts or portions of the thread remain always equal to the whole ax DE, what we propoſed is evident by the preſent *Prop.* which may alſo be very elegantly deſcribed by a certain ſort of Compaſſes, a deſcription whereof *Swenterus* gives us in his *Delic. Phyſico-Math. Part.* 2. *Prop.* 20. which may be alſo done by a ſort of organical Mechaniſm, by the help of two rulers moveable in the *Foci* GN and H*n* (*n* 2.) and equal to the tranſverſe ax DE, and faſtned above by a tranſverſe ruler GH equal to the diſtance of the *Foci*, as may appear from the Figure. For if the ſtyle F be moved round within the fiſſures of the croſs rulers H*n* and GN the curve thereby deſcribed will be an ellipſis from the property we have juſt now demonſtrated of it, which it hath in every point F. For the triangles HGN and NH*n*, which have one common ſide HN, and the others equal by conſtruction, are equal one to another, and conſequently the angles FHN and FNH equal, ſo alſo the legs HF and FN, and ſo likewiſe FN and F*n* together are equal to H*n* = DE; which is the very property of the ellipſe we are now treating of. But *Van Schooten*, who taught us this delineation, hints, that, if thro' the middle point I of the line HN you draw the line IFL, it will touch the ellipſis in the point F; for ſince the angles IFH and IFN are equal, by what we have juſt now ſaid, the vertical angle LF*n* of the one IFH, will be neceſſarily equal to the other IFN: But this equality of the angles, made by the line KL drawn thro' F, with both thoſe drawn from the centres, is here a ſign of contact, as is in the circle the equality of the angles with a line drawn from its one centre. So that after this way you may draw a tangent thro' any given point F of the ellipſis without this organical *apparatus* of Rulers; *viz.* if, having drawn from both the *Focus's* thro' the given point F the right lines *n*H, NG equal to the *Latus Tranſverſum* DE, you biſect HN in I and draw IFL: Or if the line that connects the extremes GH be produced to K, and you draw thence KFL, *viz.* in that caſe where GH and N*n* are not parallel; otherwiſe a line drawn thro' the point F parallel to them would be the tangent ſought.

As to the hyperbola, there is a mechanick method of drawing that alſo, not unlike the others, from a like property in that, communicated by the ſame *Van Schooten*, *viz.* If having found the *Focus's* N and *n* (*Fig.* 129. *n.* 3.) you tye a thread NFO in the *Focus* N and at the end of the ruler *n*O of the length of the tranſverſe ax DE; then putting in a pen or the moveable leg of a pair of compaſſes (nor would it be difficult to accommodate the practice we before made uſe of to this alſo) draw or move it within the thread NFO from O to E, ſo that the part of the thread NO may always keep cloſe to the ruler as if it were glued to it. For if we call the length of the thread X, and the tranſverſe ax *ob* as above, the ruler *n*O will be, by the Hypoth. $=$ X † *ob*. Make now the part of the thread OF $= \frac{1}{2}$ X, the remainder or other part will be NF $= \frac{1}{2}$ X and *n*F $= \frac{1}{2}$ X † *ob*, and the difference between FN and F*n*, $=$ *ob*. Make OF $= \frac{3}{4}$ X, then will FN be $\frac{1}{2}$ X and F*n* $\frac{1}{4}$ X † *ob*, the difference ſtill remaining *ob* and ſo *ad infinitum*. In ſhort, ſince the difference of the whole thread and of the whole ruler is *ob*, and in drawing them, the ſame OF is taken from both, there will always be the ſame difference of the remainders. Hence alſo aſſuming at pleaſure the points N and *n* you may deſcribe hyperbola's ſo, the thread NFO be ſhorter than the ruler *n*FO: For if it were equal there would be deſcribed a right line perpendicular to N*n*, thro' the middle point C.

There yet remains one method of deſcribing hyperbola's and ellipſes in *Plano*, by finding the ſeveral points without the help or *Apparatus* of any threads or inſtruments, *viz.* in the ellipſis, having given or aſſumed the tranſverſe axis DE and the *Foci* N and *n* (*Fig.* 130. *n.* 1.) if from N at any arbitrary diſtance, but not greater than half the tranſverſe ax NF, you make an arch, and keeping the ſame opening of the compaſſes

paſſes you cut off, from the tranſverſe ax, EG, and then, taking the remaining interval GD, from *n* you make another arch cutting the former in F, and ſo you will have one point of the ellipſe, and after the ſame way you may have innumerable others, *f*, *f*, *f*, *&c.*

In like manner to delineate the hyperbola, having given or aſſumed the tranſverſe ax DE and the *Focus's* N and *n* (*n.* 2.) if from N at any arbitrary diſtance NF you ſtrike an arch, and keeping the ſame aperture of the compaſſes from the diameter continued, you cut off EG, and then at the interval GD from *n* make another arch cutting the former in F, you will have one point of the hyperbola, and after the ſame way innumerable others, *f*, *f*, *&c.*

Propoſition IX.

If the ſecondary ax, or conjugate diameter of the hyperbola AB (Fig. 131.*) be applied parallel to the vertex E, ſo that it may touch the hyperbola, and OE, EP are equal like BC and AC, and from the centre C you draw thro' O and P right lines running on* ad infinitum, *and laſtly QR parallel to the Tangent OP; you'l have the following*

CONSECTARYS.

I. THE parts QG and HR intercepted between the curve and thoſe right lines CQ, CR will be equal; for by reaſon of the ſimilitude of the △ △ CEP and CFR as alſo CEO, CFQ as CE is to EO (and EP) ſo will CF be to FQ and FR, and conſequently theſe will be equal; and ſo taking away the ſemiordinates FG and FH which are alſo equal, the remainders GQ and HR will be alſo equal, and conſequently the □ □ QGR, GRH, *&c.* all equal among themſelves: Which we had already deduced before in *Conſect.* 2. and 3. *Def.* 7.

II. The rectangle QGR will be = □ EO or EP = $\frac{oocd}{4}$ i.e. (as *Apollonius* ſpeaks) to the fourth part of the figure: For by reaſon of the ſimilitude of the △ △ CEO, CFQ, CE will be to EO as CF to FQ: *i. e.*

as the □ CE to the □ EO
i. e. as (by *Conſ* 2. 7.) the *Lat. Tranſv.* ⎫ ſo the □ CF
to the *Lat. Rect.* ⎬ to the □ FQ
i. e. (by the 7. *Prop.*) as the ▭ DFE to the □ FG ⎭

But now if from the □ CF you take the ▭ DFE, there will remain the □ CE, by *Prop.* 9. *lib.* 1. and if from the □ FQ you take the ▭ FG there will remain the ▭ QGR, by *Prop.* 8. *lib.* 1. wherefore that remaining □ CE to this remaining ▭ QGR will be, as was the whole ſquare CF to the whole ſquare FQ, by *Prop.* 26. *lib.* 1. *i. e.* as was the □ CE to the □ EO; conſequently the ▭ QGR and the ſquare EO (to which the ſame ſquare CE bears the ſame proportion) will be equal among themſelves.

III. Since this is alſo after the ſame manner certain of any other rectangle *ggr* or *grh*, *&c.* it follows that all ſuch rectangles are equal among themſelves.

IV. Wherefore it is moſt evident, ſince the lines FR, *fr*, *&c.* and ſo GR and *gr* grow ſo much the longer, by how much the more remote they are from the vertex E; that on the contrary the lines QG and *qg* muſt neceſſarily ſo much the more decreaſe and grow ſhorter, and conſequently the right line CQ approach ſo much nearer and nearer to the curve EG.

V. But that they can never meet or coincide altho' produced *ad infinitum* will thus appear; if it were poſſible there could be any concourſe or meeting, ſo that the point G and Q or *g* and *q* could any where coincide, it would follow from *Conſect.* 2. that as the ▭ DFE to the ſquare FG ſo the ſquare CF to the ſquare FQ, *i e.* to the ſame ſquare FG; and ſo that the ▭ DFE would be = □ CF; which is abſurd by *Prop.* 9 *lib.* 1. ſo that now it is evident that the lines COQ and CPR drawn according to *Conſect.* 1. are really Aſymptotes (*i. e.* they will never (α) coincide (*viz.* with the curve of the hyperbola) as *Apollonius* has named them.

(α) *Apoll. Prop.* 1. *lib.* 2.

VI. Having drawn the right lines from G and *g* parallel to both the aſymptotes, *viz.* GS and *gs* and likewiſe GT and *gt*, the rectangles TGS and *tgs* will be (α) equal among themſelves. For by reaſon of the ſimilitude of the △ △ TQG and *tqg*, *firſt*, TG will be to QG as *tg* to *qg*; and, by reaſon of the equality of the □ □ QGR and *qgr*, *ſecondly*, QG will be to *gr* reciprocally as *qg* to GR, by *Prop.* 19. *lib.* 1. and by reaſon of the ſimilitude of the △ △ SGR and *sgr*, *thirdly*, as *gr* to *gs* ſo GR to GS, wherefore (ſince in two ſeries

1. 2. 3.
as TG to QG to *gr* to *gs*
ſo *tg* to *qg* to GR to GS)

you'l have *ex æquo* or by proportion of equality as TG to *gs* ſo *tg* to GS, by *Prop.* 24 *lib.* 1. Therefore, by *Prop.* 17. of the ſame, the □ of TG into GS = □ of *tg* into *gs*. Q. E. D.

(α) *Apollon. Prop.*12. *lib.*2.

SCHOLIUM.

HEnce, laſtly, we have a new geneſis of the hyperbola in *Plano* about its given diameters from the ſpeculations of (β) *De Witt*; if, *viz.* having drawn the lines AB and EF croſs one another at pleaſure (*Fig.* 132.) to the angle BCF you conform the moveable angle BCD (*acd* being to be delineated in the oppoſite hyperbola equal to the contiguous ACD) one of whoſe legs is conceived to be indefinitely extended, but the other CD of any arbitrary length; and to the end of it D apply the ſlit of a moveable ruler GD about the point G at any arbitrary interval GD (but yet parallel to the leg CB in this firſt ſtation) and ſo carrying together along with it the moveable angle BCD about the line ECF, but ſo that the leg CD may always remain faſt to it, and the other CB be interſected in its progreſs by the ruler GDH, *e. g.* in *b* or β. This point of interſection, thus continually moved on, will deſcribe the curve *b*Gβ, which we thus prove to be an hyperbola: Becauſe the ruler GDH turning about the pole G, and carried from D *e. g.* to *d* or δ cuts the leg of the moveable angle CB brought

(β) *De Witt Elem.Curv.lib.* 1.*cap.*2.*prop.*3.

brought to the ſituation *cb* or $\gamma\beta$, and in the mean while remaining always parallel to it ſelf; and having drawn from the points of interſection *b* and β and G the lines GI, *b*K and $\beta\kappa$ parallel to the ruler CF, becauſe *e. g.* in the ſecond ſtation, having taken the common quantity *c*D from the equal ones CD and *cd*, the remainders C*c* and D*d* are equal, and by reaſon of the ſimilitude of the △ △ *dcb* and *d*DG,

$$\text{as}\ \left.\begin{array}{l}\mathrm{D}d\\ \textit{i. e.}\ \mathrm{C}c\\ \text{or}\ b\mathrm{K}\end{array}\right\}\ \text{to DG, ſo}\ \left.\begin{array}{l}dc\\ \textit{i. e.}\ \mathrm{DC}\\ \text{or GI}\end{array}\right\}\ \text{to}\ cb\,;$$

the rectangle of K*b* into *bc* will be = □ of DG into GI, by *Prop.* 18. *lib.* 1. and in like manner, when in the third ſtation having added the common line Dγ to the equal ones CD and $\gamma\delta$, the whole lines Dδ and Cγ are equal, and, by reaſon of the ſimilitude of the △ △ $\beta\gamma\delta$ and GDδ.

$$\text{as}\ \left.\begin{array}{l}\mathrm{D}\delta\\ \textit{i.e.}\ \mathrm{C}\gamma\\ \text{or}\ \beta\kappa\end{array}\right\}\ \text{is to DG ſo is}\ \left.\begin{array}{l}\gamma\delta\\ \textit{i. e.}\ \mathrm{DC}\\ \text{or GI}\end{array}\right\}\ \text{to}\ \gamma\beta\,;$$

the □ of $\kappa\beta$ into $\beta\gamma$ = □ of DG into GI, by the ſame 18. *Prop.* Wherefore the three points *b*, G, β, (and ſo all the others that may be determined the ſame way) are in the hyperbola, whoſe aſymptotes are CB and CF and its centre C, *&c.* by the preſent *Prop. Conſect.* 6. Q. E. D.

You may alſo determine innumerable points of this curve ſeparately without the motion we have now preſcrib'd, *viz.* as the point *a* in the oppoſite hyperbola, if thro' any aſſumed point *c* in the aſymptote CE you draw a parallel to the other aſymptote CA, and having made *cd* equal to CD, from G thro' *d* draw G*da*, and ſo in others.

CHAP. II.

Of Parabolical, Hyperbolical and Elliptical Spaces.

Propoſition X.

(a) *Archim. de Quadratur. Parab. Prop.* 17. *and* 24.

THE (a) *Parabolick Space* (i. e. *in Fig.* 133. *that comprehended under the right line* GH *and the parabola* GEH) *is to a circumſcribing Parallelogram* GK, *as* 4 *to* 6 (*or* 2 *to* 3) *but to an inſcribed* △ GEH *as* 4 *to* 3.

Demonſtration.

Suppoſe FH divided firſt into two then into four equal parts, and draw parallel to the ax EF the lines *ef*, *ef*, *&c.* dividing alſo EF into four parts, the firſt *fg* will be 3, the ſecond 2, the third 1, by *Prop.* 34. *lib.* 1. but as *ef* is to *ge* ſo is *ge* to *he*, by *Conſect.* 1. *Prop.* 4. Therefore *he* in the diameter EF is $= 0$, in the firſt *ef* it is $= \frac{1}{4}$ (for as *ef*, 4, to *ge*, 1, ſo *ge*, 1, to *he*, $\frac{1}{4}$) in the ſecond *ef* a portion of *he* is $= \frac{4}{4}$, in the third to $\frac{9}{4}$, and ſo the portions *eh* in the trilinear figure E*h*HK make a ſeries in a duplicate arithmetical progreſſion, *viz.* 1, 4, 9, 16: After the ſame manner, if the parts F*f*, *&c.* are biſected, you'l find the portions *eh* in the external trilinear figure to make this ſeries of numbers $\frac{1}{8}$, $\frac{4}{8}$, $\frac{9}{8}$, $\frac{16}{8}$, $\frac{25}{8}$, $\frac{36}{8}$, $\frac{49}{8}$, $\frac{64}{8}$, and ſo onwards. Wherefore ſince the portions *eh* or the indiviſibles of the trilinear ſpace circumſcribed about the parabola are always in a duplicate arithmetical progreſſion: the ſum of them all will be to the ſum of as many indiviſibles of the parallelogram FK, equal to the line KH, *i. e.* the trilinear ſpace it ſelf to this parallelogram as 1 to 3, by *Conſect.* 10. *Prop.* 21. *lib.* 1. Wherefore the ſemi-parabola FE*h*H will be as 2, and the △ FEH as 1 $\frac{1}{2}$; therefore the whole parabola as 4, and the whole △ GEH as 3, and the whole parallelogram GK as 6. Q. E. D.

CONSECTARY I.

IT is evident (α) that in the firſt diviſion, the ſecond line *fh* (*i. e.* that drawn from the middle of the baſe FH) is three ſuch parts whereof FE is 4; for *eh* is $\frac{4}{4}$ *i. e.* 1, therefore *fh* is 3.

(α) *Archim. Prop.* 19. *with the Coroll.*

CONSECTARY II.

IT is alſo evident, that this demonſtration will hold of any parabolick ſegment.

Propoſition XI.

THE Elliptical Space (α) *comprehended by the Ellipſis DAEB* (*Fig.* 127.) *is to a circle deſcribed on the tranſverſe ax DE, as the Axis Rectus or conjugate diameter AB to the tranſverſe ax DE.*

(α) *Archim. lib. de Conoid.* &c. *Prop.* 5.

Demonſtration.

THis is in the firſt place evident from the geneſis of the ellipſe we deduced in *Schol.* 1. *Prop.* 7. for in that deduction we ſhewed that FO, *i. e.* HN was to NI as AB to DE: Which ſince it is true of all the other indiviſibles or ordinates HN and IN *ad infinitum*; it is manifeſt that the planes themſelves conſtituted of theſe indiviſibles will have the ſame reaſon among themſelves, as the *Axis Rectus* AB to the tranſverſe DE. Q. E. D.

CONSECTARY I.

THerefore the quadrature of the ellipſe will be evident, if that of the circle be demonſtrated.

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CONSECTARY II.

SInce a circle deſcribed on the leaſt diameter AB will be to one deſcribed on the greater diameter DE, as AB to a third proportional by *Prop.* 35. *lib.* 1. it follows by vertue of the preſent *Prop.* that the ellipſe is a mean proportional between the greater and leſſer circle, *i e.* as the ellipſe is to the greater circle ſo is the leſſer circle to it, *viz.* as AB to DE.

CONSECTARY III.

HEnce you may have a double method of determining the area of an ellipſe. 1. If having found the area of the greater circle, you ſhould infer, as the greater diameter of the ellipſis to the leſs, ſo the area of the circle found to the area of the ellipſe ſought. 2. If having alſo found the area of the leſſer circle, you find a mean proportional between that and the area of the greater.

SCHOLIUM.

WE may alſo ſhew the laſt part of the ſecond *Conſect.* thus, 1. If having deſcribed the circle E*adb*E (*Fig.* 134.) about the leaſt axis of the ellipſe we conceive a regular hexagon to be inſcribed, and an ellipſe coinciding with one end E of its tranſverſe ax, and with the other or oppoſite one D to be ſo elevated, that with the point *d* it may perpendicularly hang over the circle, and further from all the angles of the figure inſcribed in the circle you erect the perpendiculars gG, *b*B, *&c.* it is certain that the ſides ED and E*d* of the triangle DE*d* will be cut by the parallel planes FG*gf*, *&c.* into proportional parts, and that thoſe by reaſon of the ſimilitude of the △ △ FDG and *fdg*, and ſo alſo the other rectangles will be among themſelves as the intercepted parts of the lines ID and *id*, CI and *ci*, and *in infinitum*, (*viz.* of how many ſides ſoever the inſcribed figure conſiſts:) Wherefore alſo all the parts of the ellipſe taken together will be to all the parts of the circle taken together, *i. e.* the whole ellipſe to the whole circle

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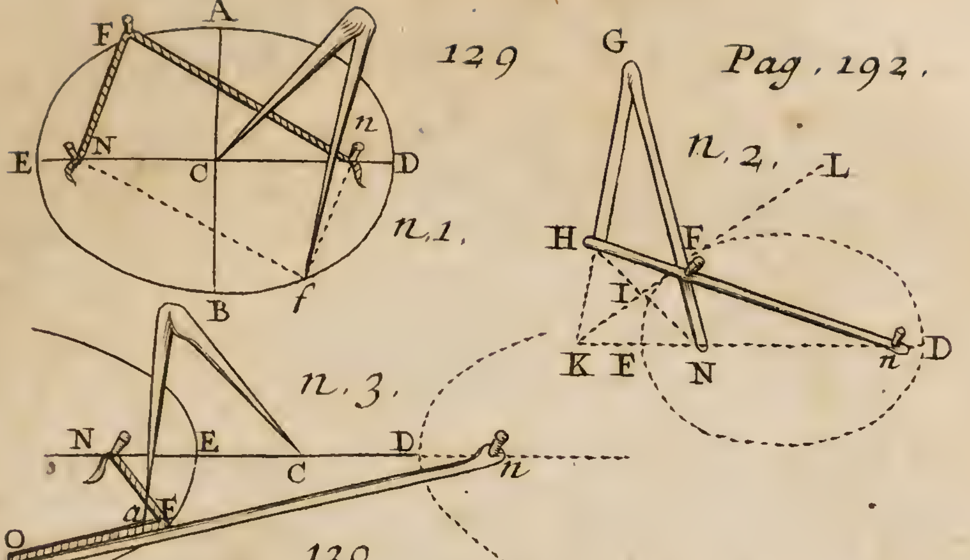

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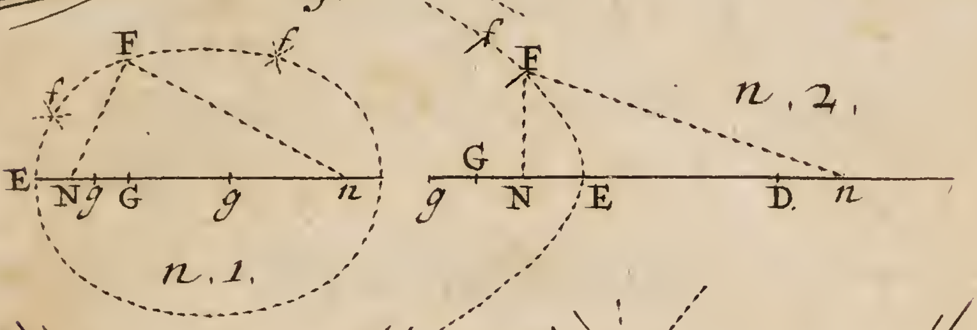

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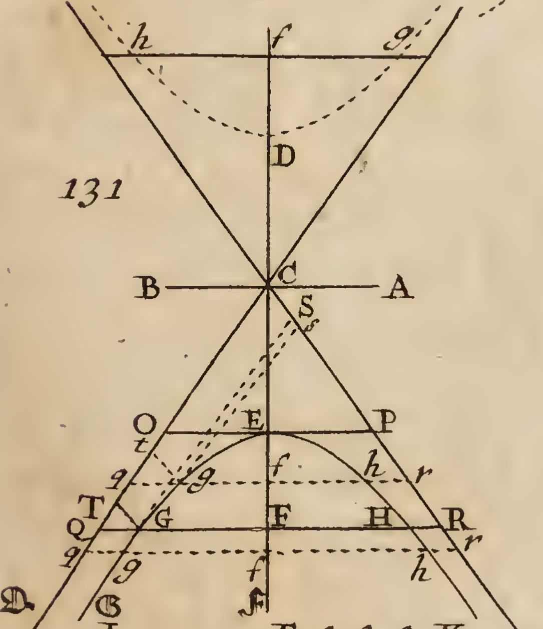

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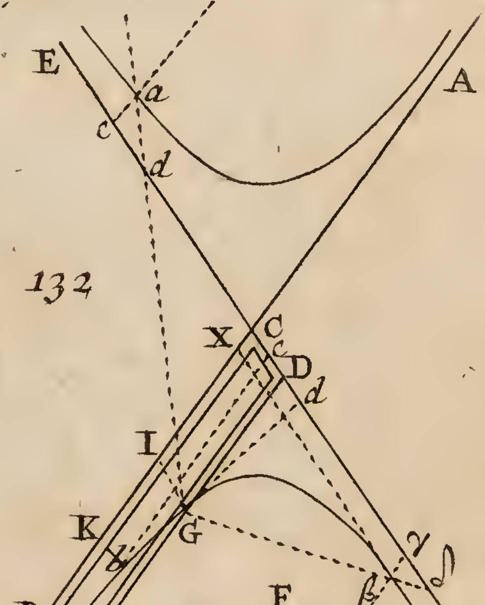

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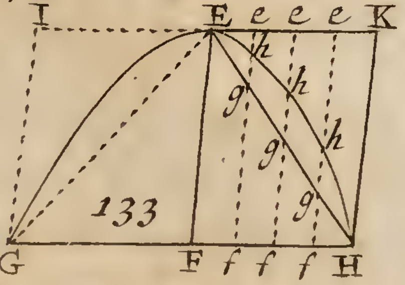

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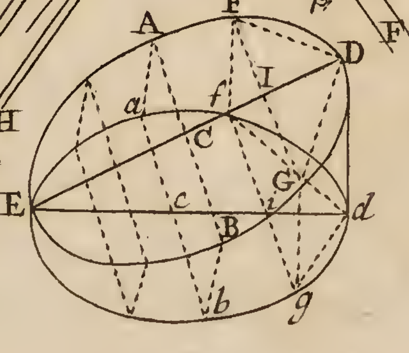

circle as all the parts of the diameter ED or *ab*, *i. e.* as DE it ſelf to AB. Q. E D.

CONSECTARY IV.

IT is alſo evident that both theſe demonſtrations of the preſent *Prop.* will be alſo the ſame in any ſegments of the ellipſis or circle.

Propoſition XII.

ANY Hyperbolical ſpace GEHG (Fig. 135) is to any Hyperbolick figure of equal heighth gEhg [whoſe Latus Rectum and Tranſverſum are equal (as in the circle) and alſo equal to the Latus Tranſverſum of the former DE, as the Axis Rectus (or conjugate) AB is to the Latus Tranſverſum DE (as in the ellipſis.)

Demonſtration.

By the Hypoth. and *Prop.* 7. and its ſecond *Conſect.* the □ Fg is = ▭ DFE. Wherefore this ▭ DFE *i e.* the □ Fg is to the □ FG as the *Latus Tranſverſum* to the *Latus Rectum* of the hyperbola GEHG, by the ſame ſeventh *Prop. i. e.* (by *Conſect.* 2. of the ſame) as the ſquare of the *Latus Tranſverſ.* DE to the ſquare of the conjugate AB : Therefore the roots of theſe ſquares will be alſo proportional, *viz.* Fg to FG as DE to AB ; and conſequently (ſince the ſame is true of any other ordinates *ad infinitum*) the whole hyperbola *g*E*hg* will be to the whole one GEHG as DE to AB. Q. E. D.

CONSECTARY I.

THerefore having found the quadrature of ſuch an hyperbola, whoſe *Latus Rectum* and *Tranſverſum* are equal, you may have alſo the quadrature of any other hyperbola.

CONSECTARY II.

IT is evident that the ſame demonſtration will hold in any other hyperbola's.

Propoſition XIII.

ANY Parabolick ſegments upon the ſame baſe, and hyperbolical and elliptical ones deſcribed about the ſame conjugate (one whereof ſhall be a right *one, the other a* ſcalene*) and conſtitued between the ſame parallels, are equal.*

Demonſtration.

I. It is evident of Parabola's; for both the right one GEHG, and the ſcalene one G𝔈HG (*Fig.* 136. *n.* 1.) (for the demonſtration of *Prop.* 10. will hold in both) is to a △ inſcribed in them as 4 to 3. But the triangles GEH and G𝔈H are equal, by *Conſect.* 5. *Def.* 12. or *Prcp.* 28. *lib.* 1. Therefore the Parabola's alſo.

Or thus, in the right parabola GEHG every thing is the ſame as in 1. and 4. *Prop.* of this Book, *viz.* EI = *eb*, EF = *ib*, the ſquare IK = *oecc*, the □ FG = *oicc*. And becauſe therefore in the ſcalene Parabola alſo the ſquare FG remains = *oicc*, make F𝔈 = *n*, and find both the abſciſſa 𝔈𝔍, and the □ anſwering to it 𝔍𝔎.

1. For the abſciſſa; as FE to EI ſo F𝔈 to 𝔈𝔍, *per*

$$ib \text{ —— } eb \text{ —— } n \text{ —— } \frac{en}{i}$$

Conſect. 4 *Prop.* 34 *lib.* 1.

2. For the □ 𝔍𝔎; as F𝔈 to 𝔈𝔍 ſo □ FG to □ 𝔍𝔎,

$$n \text{ —— } \frac{en}{i} \text{ —— } oicc \text{ —— F. } oecc.$$

per Prop. 4. of this.

Therefore the □ 𝔍𝔎 = □ IK and 𝔍𝔎 = IK, and this in any caſe *ad infinitum*: Therefore the one parabola is = to the other. Q. E. D.

II. The buſineſs is much after the ſame way evident of ellipſes and hyperbolas. For making all things in the ellipſis and right hyperbola (*n.* 2. and 3. *Fig.* 136.) as in *Prop.* 2, 3, 5, 7. *viz.* the □ IK *oecd* — *eecd* in the ellipſis, *oecd* + *eecd* in the hyperbola, the □ AB *oocd* by *Conſect.* 2. *Prop.* 7. EI = *eb*, DE = *ob*, *&c.* if in oblique ones for the *Latus Tranſverſum* $\mathfrak{DE}$ you put *n*, and ſeek the *Latus Rectum* and abſciſſa $\mathfrak{EI}$, you may by means of theſe alſo have the ſquare $\mathfrak{IK}$, by *Prop.* 2. and 3.

1. *For the Latus Rectum.*

As n to $\sqrt{oocd}$ ſo $\sqrt{oocd}$ to $\frac{oocd}{n}$, by *Conſ.* 2. 7:

2. *For* $\mathfrak{EI}$ *the abſciſſa.*

As ob to eb ſo n to $\frac{en}{o} = \mathfrak{EI}$.

3. *For the ſide RS* ▭ *deficient or exceeding, from* Prop. 2. *and* 3.

As n to $\frac{oocd}{n}$ ſo $\frac{en}{o}$ to $\frac{ocde}{n}$ = RS.

Now the abſciſſa multiplyed by the *Lat. Rect.*

$\frac{en}{o}$ by $\frac{oocd}{n}$ gives ▭ *oecd.*

The ſum of theſe ▭ ▭ *oecd* + *eecd* in the hyperb. = □ $\mathfrak{IK}$ by *Prop.* 2. evidently = □ IK.

The abſciſſa multiplyed by RS.

$\frac{en}{o}$ by $\frac{ocde}{n}$ gives ▭ *eecd.*

The difference of theſe ▭ ▭ *oecd* — *eecd* gives in the ellipſis □ $\mathfrak{IK}$ by *Prop.* 3. evidently = □ IK.

Wherefore the lines $\mathfrak{IK}$ and IK, and the whole $\mathfrak{KL}$ and KL will be equal; and ſince the ſame thing is evident after the ſame way of all other lines of this kind *ad infinitum*, the elliptical and hyperbolical ſegments will be ſo alſo. Q. E. D.

CHAP. III.

Of Conoids and Spheroids.

Propoſition XIV.

A Parablick Conoid (α) *is ſubduple of a Cylinder, and in ſeſquialteran reaſon* (*or as* $1\frac{1}{2}$) *of a cone of the ſame baſe and altitude.*

(α) *Archimed. Prop* 23. *and* 24. (*al.* 26. *and* 27.)

Demonſtration.

Becauſe in the parabola the □ AD (*Fig.* 137.) is to the □ SH, as BD to BH, *i. e.* as 3 to 1, and ſo to the □ TI as BD to BI, *i. e.* as 3 to 2, by *Prop.* 4. of this; it is evident that theſe ſquares of SH and TI and AD and conſequently of the whole lines alſo S*h*, T*i*, AC, and the circles anſwering to them will be in arithmetical progreſſion, 1, 2, 3; and moreover if there are new Biſections *in infinitum*, as the abſciſſa's ſo alſo the ſquares and circles of the ordinates, by vertue of the aforeſaid fourth *Prop.* will always be in arithmetical Progreſſion 1, 2, 3, 4, 5, 6, *&c.* It is evident that an infinite ſeries of circles in the conoid, conſider'd as its indiviſibles, will be to a ſeries of as many circles equal to the greateſt AC, *i. e.* the conoid to the cylinder AF as 1 to 2, or as $1\frac{1}{2}$ to 3, by *Conſect.* 9. *Prop.* 21. or *Conſect.* 4. *Prop.* 16. *lib.* 1. but to the ſame cylinder AF the inſcribed cone ABC is as 1 to 3, by *Prop.* 38. *lib.* 1. therefore the cylinder, conoid and cone are as 3, $1\frac{1}{2}$, and 1. Q. E. D.

Propoſition XV.

THE half of (α) *any Spheroid, or any other ſegment of it is in ſubſeſquialteran proportion to the cylinder, and double of the cone having the ſame baſe and altitude.*

(α) *Archim.* 29. *and* 30. (*al.* 32. *and* 33.)

Demonſtration.

Having divided the altitude BD (*Fig.* 138) *e. g.* into three equal parts, becauſe in the ellipſe as well as in the circle the ſquare of AD is to the ſquare of SH as the □ GDB to the □ GHB, *i. e.* as 9 to 5, and ſo to the ſquare TI as 9 to 8, by *Conſect.* 1. *Prop.* 5. of this; and in like manner if you make new biſections, the ſquares (and conſequently the circles) of the ordinates go on or decreaſe by a progreſſion of odd numbers, as 36, 35, 32, 27, 20, 11, and ſo *ad infinitum*, the biſections being continued on; as we have ſhewn in the ſphere and circumſcribed cylinder *Prop.* 39. *lib.* 1. and it will neceſſarily follow here alſo (by vertue of *Conſect.* 12. *Prop.* 21.) that the whole cylinder will be to the inſcribed ſegment of the ſpheroid, as 3 to 2; and ſince the ſame cylinder is to the cone ABC as 3 to 1, alſo the ſegment of the ſpheroid will be to the cone as 2 to 1. Q. E. D.

Propoſition XVI.

AN hyperbolical Conoid (a) is to a cone of the ſame baſe and altitude, as the aggregate of the ax of the hyperbola that forms it and half the Latus Tranſverſum, to the aggregate of the ſaid axis and Latus Tranſverſum.

(a) Archim. Prop. 27. and 28. (al. 30. and 31.)

Demonſtration, containing alſo the Invention of this Proportion.

Make (in *Fig.* 139.) CE $= a$, EF $= b$, OE $= c$; then will CF $= a + b$. Since therefore

as CE to OE ſo CF to FQ

$$a \text{ —— } c \text{ —— } a + b \text{ —— } \frac{ac + bc}{a}$$

the □ EO will $= cc$ and □ FQ $= \frac{aacc + 2abcc + bbcc.}{aa}$

But as theſe ſquares ſo alſo are the circles of the lines EO and FQ to one another, by *Prop.* 32. *lib.* 1. and ſo the cone COP will be as $\frac{acc}{3}$, and the cone CQR as $\frac{acc}{3} + bcc + \frac{bbcc}{a} + \frac{b^3cc}{3aa}$ (*viz.* by multiplying the third part of the altitude CF by the baſe FQ:) Having therefore ſubſtracted the cone COP from the cone CQR, there will remain the truncated cone QOPR $bcc + \frac{bbcc}{a} + \frac{b^3cc}{3aa}$, and from this ſolid truncated cone having further ſubſtracted the hollow truncated cone, which the ſpace EHRP produced in the geneſis of the conoid (and which according to *Conſect.* 2. *Definit.* 9. is as *bcc*) there will remain the hyperbolical conoid $\frac{bbcc}{a} + \frac{b^3cc}{3aa}$, i. e. (by ſubſtituting now the values of the ax or abſciſſa EF, and of half the *Lat. Tranſv.* EC, and of the conjugate diam. OP, *&c.* found in the demonſtrations of the preceding Chapter, *viz.* $\frac{ob}{2}$ for a, eb for b, and $\sqrt{oocd}$ for c or $oocd$ for cc) the hyperbolical conoid will come out $2eebocd + \frac{4e^3bcd}{3}$ i. e. $\frac{6eebocd + 4e^3bcd}{3}$. But the cone GEH (multiplying the third part of EF into the □ GH, *i. e.* $\frac{1}{3}eb$ into $4oecd + 4eecd$) is as $\frac{4eebocd + 4e^3bcd}{3}$. Therefore the conoid is to the cone as $6eebocd + 4e^3bcd$ to $4eebocd + 4e^3bcd$, i. e. (dividing on both ſides by $4eecd$) as $\frac{3}{2}ob + eb$ to $ob + eb$. Which was to be found and demonſtrated.

SCHOLIUM.

IF any one had rather proceed herein by indiviſibles, as in the precedent Prop. having divided the ax EF (*Fig.* 140.) again into three equal parts, and aſſuming the values of the lines determined in the hyperbola, *viz.* *eb* for the abſciſſa EF, *ob* for the tranſverſe ax, $\frac{ocd}{b}$ for the *Latus Rectum*, $oecd + eecd$ for the ſquare of the ſemiordinate FG, *&c.* the loweſt and greateſt circle of the diameter HG will be as $oecd + eecd$, and, if you make

as the *Latus Tranſv.* to the *Latus Rectum*, ſo the □ D*f*E
ob —— $\frac{ocd}{b}$
made of $ob \dagger \frac{2}{3}eb$ into $\frac{2}{3}eb$ (i. e. $\frac{2}{3}oebb \dagger \frac{4}{9}eebb$) to a fourth; there will come out $\frac{2}{3}oecd \dagger \frac{4}{9}eecd$ for the ſecond circle of the diam. *hg*; and by the ſame inference (as ob to $\frac{ocd}{b}$ ſo $ob \dagger \frac{1}{3}eb$ into $\frac{1}{3}eb$ to a fourth) for the third circle of the diameter ℌ𝔊 $\frac{1}{3}oecd \dagger \frac{1}{9}eecd$; ſo that theſe indiviſibles [for which here and in the precedent alſo the partial circumſcribed cylinders may be aſſumed] proceed in a double ſeries of numbers, the firſt in a ſimple arithmetical progreſſion 3, 2, 1, the latter in a duplicate Arithmetical progreſſion of ſquares 9, 4, 1; and the ſame if you make further new biſections, will neceſſarily happen *ad Infinitum*, (the former numbers *e. g* in the firſt biſection will be $\frac{6}{6}\,\frac{5}{6}\,\frac{4}{6}\,\frac{3}{6}\,\frac{2}{6}\,\frac{1}{6}\,oecd$ the latter $\frac{36}{36}\,\frac{25}{36}\,\frac{16}{36}\,\frac{9}{36}\,\frac{4}{36}\,\frac{1}{36}\,eecd$, &c) it is manifeſt from the conſectaries of *Prop.* 21. *lib.* 1. that the whole cylinder HK will in like manner be expreſſed by a double ſeries of parts anſwering, in numbers to the indiviſibles of the conoid made by any biſection, but in magnitude to the greateſt of them all, and in the ſum of its firſt ſeries of parts will be to the ſum of the firſt in the conoid, both being infinite, as 2 to 1 or 3 to $1\frac{1}{2}\,oecd$, by *Conſect.* 9 of the ſaid *Prop.* 21. and the ſum of its latter to the ſum of the former in the conoid will be as 3 to 1 $eecd$ and ſo the whole cylinder to the whole conoid as $3\,oecd \dagger 3\,eecd$ to $1\frac{1}{2}\,oecd \dagger eecd$ i. e. (dividing by ecd) as $3\,o \dagger 3\,e$ to $1\frac{1}{2}\,o \dagger e$ i. e. multiplying both ſides by b) as $3\,ob \dagger 3\,eb$ to $1\frac{1}{2}\,ob \dagger eb$; and conſequently the cone (which is $\frac{1}{3}$ of the cylinder) to the conoid as $ob \dagger eb$ to $1\frac{1}{2}ob \dagger eb$. Q. E. D.

CONSECTARY.

HEnce alſo appears the proportion of the hyperbolick conoid to a cylinder of the ſame baſe and altitude, which we did not expreſs in the *Prop.* *viz.* as the aggregate of the ax and half the *Latus Tranſverſum* to triple the aggregate of the ſaid ax and *Latus Tranſverſum.*

CHAP.

CHAP. IV.

Of Spiral Lines and Spaces.

Propoſition XVII.

(α) *Archim. Prop.* 24. *de Spiral.*

T*HE* (α) *firſt ſpiral ſpace is ſubtriple of the firſt circle,* i. e. *as* 1 *to* 3.

Demonſtration.

Having divided the circumference of the circle into (*Fig.* 141. *n.* 1.) three equal parts by lines drawn from the initial point, beginning from the firſt line BA, the line BC will be as 1, BD as 2, BA as 3, by *Conſect.* 1. *Def.* 12. of this book, and conſequently the ſectors circumſcribed about the ſpiral will be CB*c* as 1, DB*d* as 4, AB*a* as 9, by *Prop.* 32. *lib.* 1. and in like manner, if you make new biſections, the lines drawn from the point B to the ſpiral, will be 1, 2, 3, 4, 5, 6; but the circumſcrib'd ſectors, 1, 4, 9, 16, 25, 36; and ſo the circumſcrib'd partial ſectors *ad infinitum* will proceed in an order of ſquares, there being always as many ſectors in the circle equal to the greateſt of them. Therefore all the ſectors that can be circumſcrib'd *ad infinitum* about the ſpiral ſpace, *i. e.* the ſpiral ſpace it ſelf (in which at laſt they end) to ſo many equal to the greateſt, *i. e.* to the circle, is as 1 to 3, by *Conſect.* 10. *Prop.* 21. *lib.* 1. Q. E. D.

CONSECTARY I.

Since the firſt circle is to the ſecond as 1 to 4 (*i. e.* as 3 to 12) by *Def.* 12. of this, and *Prop.* 31. *lib.* 1. and the firſt ſpiral ſpace to the firſt circle as 1 to 3 by the preſent Prop. the ſame ſpiral ſpace will be to the ſecond circle as 1 to 12; and to the third by a like inference as 1 to 27, to the fourth as 1 to 48, *&c.*

CONSECTARY II.

THE firſt ſpiral line is equal to half the circumference of the firſt circle. For the lines or radii of the ſectors, and conſequently their peripheries or arches proceed in a ſimple arithmetical reaſon, as 1, 2, 3, 4, 5, 6, *&c.* while in the mean time the whole periphery of the circle contains ſo many arches equal to the greateſt. Therefore the whole periphery of the circle is to an infinite ſeries of circumſcrib'd arches, *i. e.* to the ſpiral line it ſelf, as 2 to 1, by *Conſect.* 9. *Prop.* 21. *lib.* 1.

Propoſition XVIII.

THE whole ſpiral (α) ſpace comprehended under the ſecond right line EA and the ſecond ſpiral EGIA (ſee Fig. 141. *n.* 2.*) is to the ſecond circle as* 7. *to* 12.

(α) Archim. Prop. 25.

Demonſtration.

For having divided the circumference of the circle firſt into three equal parts, there will be drawn to the ſecond ſpiral four right lines BE, BG, BI and BA being as 3, 4, 5, 6, and but only three ſectors circumſcrib'd, *viz.* GB*g*, IB*i* and AB*a*, which proceed according to the ſquares of the three latter lines, *viz.* 16, 25, 36, ſo that the ſum is 77, while the ſum of three equal to the greateſt is 108, and ſo the one to the other (dividing both ſides by 9) as 12 to $8\frac{5}{9}$. Having moreover biſected the arches and parts of the line BE, ſo that that ſhall be 6, the ſecond BF will be 7, and ſo the other five 8, 9, 10, 11, 12; and the ſectors anſwering to them (excepting the firſt) 49, 64, 81, 100, 121, 144, ſo that their ſum ſhall be 559, while the ſum of ſix equal to the greateſt, *i. e.* the whole circle is 864, and ſo one to the other (dividing both by 72) as 12 to $7\frac{55}{72}$. In the other biſection of the arches and the parts of the line BE, ſo that the one ſhall be 12, the ſecond 13, *&c.* to the thirteenth BA which will be 24, the ſum of twelve ſectors will be found to be 4250; and the ſum of

25

as many equal to the greateſt 6912, and ſo the one to the other (dividing both ſides by 576) as 12 to $7\frac{218}{576}$ ſ. $\frac{109}{288}$. Therefore the proportion will be

I. In the firſt caſe 12 to 7 † 1 † $\frac{1}{2}$ † $\frac{1}{18}$ *viz.* $\frac{4}{72}$.
II. In the ſecond caſe 12 to 7 † $\frac{1}{2}$ † $\frac{1}{4}$ † $\frac{1}{72}$ *viz.* $\frac{4}{288}$.
III. In the third caſe 12 to 7 † $\frac{1}{4}$ † $\frac{1}{8}$ † $\frac{1}{288}$, *&c.*

The firſt and ſecond fractions thus decreaſing by $\frac{1}{2}$ the latter by $\frac{1}{4}$. Wherefore the proportion of the ſecond circle to the ſecond ſpiral ſpace will be as

12 to 7 † 1 † $\frac{1}{2}$ † $\frac{1}{18}$
—— $\frac{1}{2}$ —— $\frac{1}{4}$ —— $\frac{3}{72}$
—— $\frac{1}{4}$ *&c.* —— $\frac{1}{8}$ *&c.* —— $\frac{3}{288}$ *&c.*

By vertue of *Conſect.* 3. and 8. ⊐⊏ ○ ⊐⊏ ○
Prop. 21. *lib.* 1.
i. e. as 12 to 7. Q. E. D.

CONSECTARY I.

BEcauſe the ſecond circle is to the firſt ſpiral ſpace as 12 to 1, by *Conſect.* 1. of the preceding *Prop.* and to the ſecond ſpiral ſpace as 12 to 7, *by the preſent.* it will be to the ſecond ſpace without the firſt (*viz.* BCDEAIGE) as 12 to 6 *i. e.* as 2 to 1.

CONSECTARY II.

THerefore the ſecond ſpace ſeparately to the firſt is as 6 to 1.

CONSECTARY III.

SInce in the triſection of both theſe circles, firſt and ſecond, there ariſe ſix lines, and as many ſectors, *viz.* three lines BC, BD, BE, *i. e.* 1, 2, 3, to which there anſwer three arches in the ſame progreſſion within the ſecond circle, and alſo as many equal to its greateſt; therefore the ſum of all the unequal arches will be 21, but the ſum of the equal ones of both circles (each of which in the firſt are equivalent to 3, in the ſecond to 6) will be 27. Wherefore the ſum of both the Peripheries to the ſum of all the circumſcrib'd arches will be as

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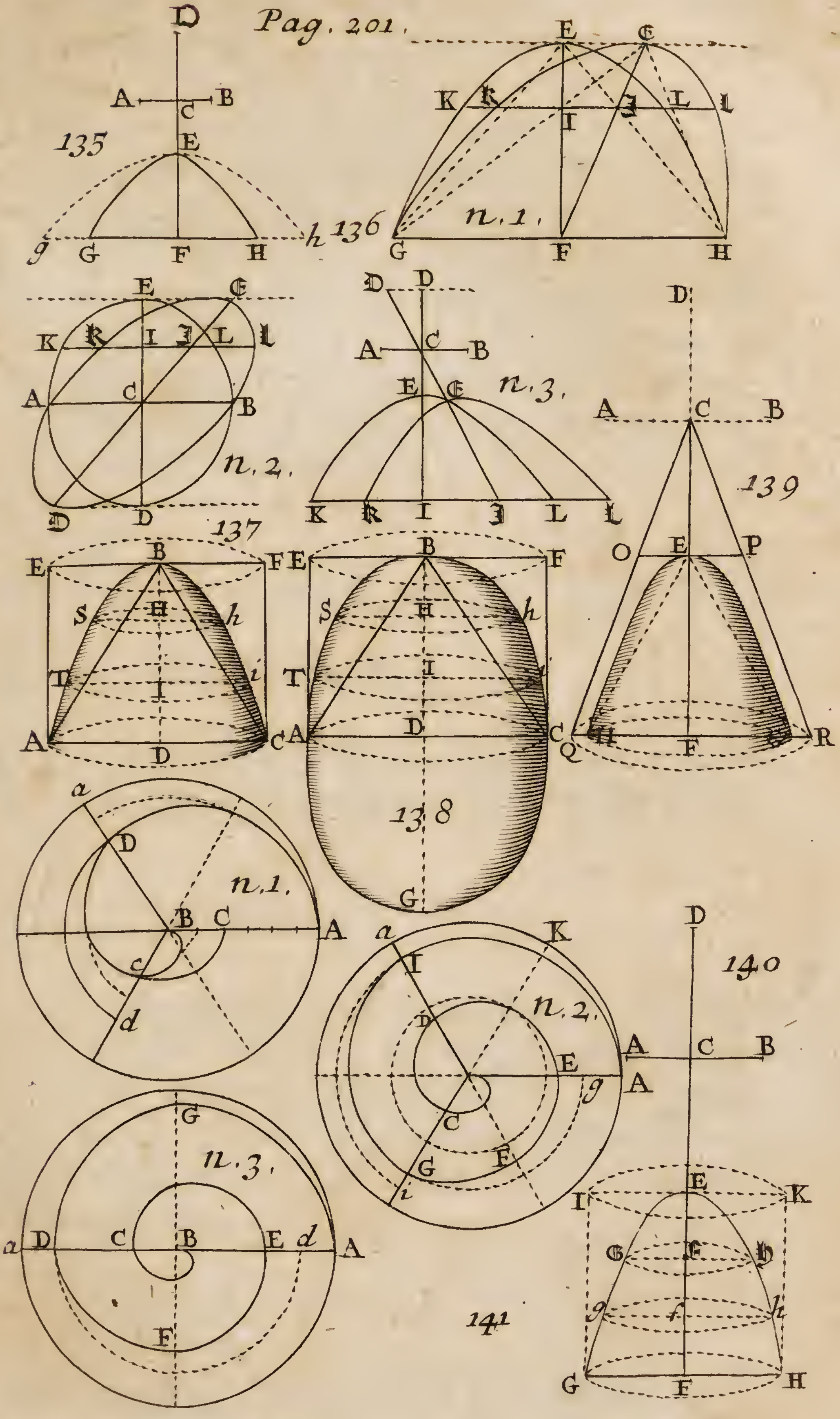
135
D
A C B
E
g G F H h
136
n.1.
E 𝔈
K 𝔎 I ℑ L 𝔏
G F H
n.2.
E 𝔈
K 𝔎 I ℑ L 𝔏
A C B
𝔇 D
n.3.
𝔇 D
A C B
E 𝔈
K 𝔎 I ℑ L 𝔏
137
E B F
S H h
T I i
A D C
138
E B F
S H h
T I i
A D C
G
139
D
A C B
O E P
Q H F G R
140
D
A C B
E
I K
𝔊 𝔉 𝔥
g f h
G F H
141
n.1.
a D B C A c d
n.2.
a I K D A E g C G F i
n.3.
G
a D C B E d A
F

27 to 21, *i. e.* (dividing both ſides by 9) as 3 to $2\frac{1}{3}$. Moreover biſecting the arches of the circles and the parts of the line BA, there will ariſe ſix circumſcribed unequal arches within the firſt circle, which are as 1, 2, 3, 4, 5, 6, and as many within the ſecond 7, 8, 9, 10, 11, 12; the ſum of all which is 78, while the ſum of as many equal ones on both ſides is 108. Wherefore the one will be to the other, *i. e.* the ſum of both the peripheries to twelve circumſcribed arches taken together, is now as 108 to 78, *i. e.* (dividing both ſides by 36) as 3 to $2\frac{1}{6}$. And making yet another biſection, the proportion will be found to be as 3 to $2\frac{1}{12}$, *&c* and hence at length may be evidently inferr'd; that the ſum of both the peripheries will be to the ſum of all the arches circumſcribible *ad infinitum*, i. e. to the whole helix as

3 to $2 + \frac{1}{3}$
$- \frac{1}{6}$
$- \frac{1}{12}$ *&c.* $= 0$. that is, as 3 to 2. Q. E. D.

CONSECTARY IV.

THerefore, ſince the periphery of the ſecond circle is double of the firſt, that alone will be equal to the whole ſpiral.

CONSECTARY V.

THerefore, if the periphery of the ſecond circle be 2, the periphery of the firſt will be 1, and the firſt ſpiral line $\frac{1}{2}$ by *Conſect.* 2. of the anteced. *Prop.* wherefore the ſecond ſpiral alone will be $1\frac{1}{2}$, and ſo the periphery of the ſecond circle alone will be to the ſecond ſpiral alone as 2 to $1\frac{1}{2}$ *i. e.* as 4 to 3; and to the firſt alone as 4 to 1.

SCHOLIUM I.

BUT as *Conſect.* 4. may be alſo deduced after another way, *viz.* by comparing only the arches of the ſecond circle with the correſpondent circumſcripts, but conſidering them as taken twice (becauſe that circle is twice turned round while the whole helix or ſpiral is deſcribed) and finding in the firſt triſection the proportion of double the ſecond periphery to all the circumſcripts as 12 to 7; and in the ſucceeding biſection as 12 to $6\frac{1}{2}$; in the ſecond biſection as 12 to $6\frac{1}{4}$, *&c.* and at length by inferring, that the ſecond periphery is double of all the arches circumſcribible about the whole helix *ad infinitum*, that is to the helix it ſelf.

as 12 to 6 † 1
——— $\frac{1}{2}$
——— $\frac{1}{4}$ *&c.* ≍ 0. *i. e.* as 12 to 6;

and conſequently the ſimple ſecond periphery will be to the whole helix as 6 to 6: Thus the 5. *Conſect.* may be ſeparately had after the ſame manner, if inſtead of the firſt triſection, you only biſect; (*vid. Fig.* 141. *n.* 3.) for ſo in the firſt biſection the arches circumſcribed about the ſecond ſpiral line would be ſeparately two ſemi-circles D*d*, 3 and A*a*, 4, (for as the line BC is one, BE, 2, BD, 3, BA, 4; ſo the arch deſcribed by the radius BD is 3 and deſcribed by the radius BA ≍ 4,) and their ſum 7; while the ſum of two equal to the greateſt is 8. In the ſecond biſection (when BE is 4) BF and its arch is made 5, the arch BD 6, the arch BG 7, the arch BA 8, the ſum 26; while the ſum of ſo many quadrants equal to the greateſt is 32. Thus in the third biſection the ſum of eight Octants circumſcrib'd about the ſecond helix will be found to be 100, the ſum of ſo many ≍ to the greateſt 128, *&c.* Wherefore the periphery of the ſecond circle in the firſt caſe will be to the arches circumſcrib'd about the ſecond helix as 4 to 3 † $\frac{1}{2}$; in the ſecond as 4 to 3 † $\frac{1}{4}$; in the third as 4 to 3 † $\frac{1}{8}$, *&c.* and ſo to all the arches circumſcribible *in infinitum*, i. e. to the ſecond helix it ſelf as

4 to 3 † $\frac{1}{2}$
——— $\frac{1}{4}$
——— $\frac{1}{8}$ *&c.* ≍ 0. *i.e.* as 4 to 3. Q. E. D.

By the ſame method you may eaſily find the proportion of the third circle to the third ſpiral ſpace, and of that periphery either to the whole ſpiral, or ſeparately to the third, as will be evident to any one who trys.

I. *For the third ſpiral ſpace.*

(*Fig.* 142)

BC	1	BF	4	BI	7	49
BD	2	BG	5	BK	8	64
BE	3	BH	6	BA	9	81

are the three firſt ſectors circumſcrib'd about the parts of the third helix. The ſum of theſe three ſectors is 194; and the ſum of ſo many equal to the greateſt 243. Therefore the firſt proportion of the one ſum to the other will be as 243 to 194, *i. e.* (dividing both ſides by 9) as 27 to 21 $\frac{1}{9}$.

In the firſt biſection there will be ſeven lines:

BH 12,	BL 13	169
	BI 14	196
	BM 15	225
	BK 16	256
	BN 17	289
	BA 18	324

Sectors circumſcribed about the parts of the third helix.

Sum 1459; while in the mean time the ſum of as many equal to the greateſt is 944, and ſo the ſecond proportion as 1944 to 1459 *i. e.* (dividing both ſides by 72) as 27 to 20 $\frac{19}{72}$.

In the ſecond Biſection there will be thirteen lines, *viz.* BH 24, the reſt 25, 26, *&c.* but the ſum of the ſectors, *i. e.* of the ſquare numbers anſwering to the twelve latter will be found to be 11306; while in the mean time the ſum of as many equal to the greateſt will be 15552, ſo that you will have the third proportion of this ſum to the other, *viz.* as 15552 to 11306, *i. e.* (dividing both ſides by 576) as 27 to 19 $\frac{362}{576}$.

Therefore the I. proportion will be as 27 to $19 \dagger 2 \dagger \frac{5}{9}$ *i. e.*
to $19 \dagger 2 \dagger \frac{1}{2} \dagger \frac{1}{18}$
II. ——— as 27 to $19 \dagger 1 \dagger \frac{19}{27}$ *i. e.*
to $19 \dagger 1 \dagger \frac{1}{4} \dagger \frac{1}{72}$
III. ——— as 27 to $19 \dagger \frac{362}{576}$ —— *i. e.*
to $19 \dagger \frac{1}{2} \dagger \frac{1}{8} \dagger \frac{1}{288}$.

Therefore the proportion of the third circle to the third ſpiral ſpace will be as 27 to 19 † 2 † $\frac{1}{2}$ † $\frac{1}{18}$

—— 1 — $\frac{1}{4}$ — $\frac{3}{72}$ *i. e.* as 27 to 19.

—— $\frac{1}{2}$ &c. —— $\frac{1}{8}$ &c. —— $\frac{3}{288}$ &c.

$= 0.$ $= 0.$ $= 0.$ Q. E. D.

II. *For the third ſpiral line.*

If inſtead of the firſt triſection (as leſs commodious for the end propoſed) you make uſe here alſo, as before, of biſection in the ſame figure, there will come out ſix lines from the point B to the helix, *viz.* B*m*, 1, BE, 2, B*n*, 3, BH, 4, B*o*, 5, BA, 6; to which there anſwer as many ſemicircular arches in the ſame progreſſion, and to the greateſt of the two as many equal to 2, 4, 6; ſo that the ſum of the unequal ones is 21, and of the equal ones 24, and ſo the proportion of three peripheries together to all the circumſcripts together will be as 24 to 21 (and dividing both by 6) as 4 to $3\frac{1}{2}$. In the ſecond biſection the twelve unequal lines and arches make the ſum 78, and as many equal to the greateſt of the four will give the ſum 96; ſo that the ſecond proportion will be 96 to 78, *i. e.* (dividing both ſides by 24) 4 to $3\frac{1}{4}$. In the third biſection the proportion will come out as 384 to 300, *i. e.* (dividing both ſides by 96) as 4 to $3\frac{1}{8}$, *&c.* Therefore the proportion of the three circles together to the whole Helix will be as 4 to 3 † $\frac{1}{2}$

—— $\frac{1}{4}$

—— $\frac{1}{8}$ *&c.* $= 0.$ *i. e.* as 4 to 3 or 12 to 9.

Q. E. D.

CONSECTARY VI.

NOW, if the periphery of the firſt circle be made 2, the ſecond will be 4, the third 6, and conſequently the ſum 12; it will be manifeſt that the third periphery ſeparately will be to the whole helix as [illegible] *e.* as 2 to 3.

CONSECTARY VII.

AND becauſe the ſecond periphery (which is 4) is equal to the firſt and ſecond helix together, by the above *Conſect.* 4. the remaining third ſpiral will be 5, and ſo the proportion of the third periphery to it as 6 to 5.

CONSECTARY VIII.

WHerefore the proportions of each of the peripheries to their correſpondent ſpirals will be in a progreſſion of ordinal numbers, *viz.* ſo that the latter of every two will denote the periphery of a circle, and the former an inſcribed ſpiral; and conſequently the ſpiral lines will be in an arithmetical progreſſion of odd numbers, and the peripheries of the circles in a progreſſion of even ones.

1———The firſt Spiral,
2———The firſt Periphery,
3———The ſecond Spiral,
4———The ſecond Periphery,
5———The third Spiral,
6 *&c.*—The third Periphery, *&c.*

SHCOLIVM II.

THE ſeventh Conſectary may alſo be eaſily deduced ſeparately this way: In the firſt biſection the line BA and its periphery is 6, the line B*o* and its periphery 5, the ſum of the circumſcribed Peripheries 11; the ſum of as many equal to the greateſt 12. Therefore the periphery of the third circle will be to the two circumſcripts as 12 to 11, *i. e.* as 6 to $5\frac{1}{2}$. In the ſecond biſection the four circumſcribed quadrants will be 12, 11, 10, 9, their ſum 42; and the ſum of four equal to the greateſt, *i. e.* the periphery of the third circle 48. Therefore the proportion is now as 48 to 42, *i. e.* (dividing both ſides by 8) as 6 to $5\frac{1}{4}$. Thus you will have the third proportion as 192 to 164, *i. e.* (dividing both ſides by 32) as 6 to $5\frac{1}{8}$. Wherefore the proportion of the third periphery to the third helix or ſpiral is

as 6 to $5 + \frac{1}{2}$
$- \frac{1}{4}$
$- \frac{1}{8}$ *&c.*
$= 0$. *i.e.* as 6 to 5. Q. E. D.

CONSECTARY IX.

AS Conſect. 8. ſupplies us with a rule to determine the proportion of every ſpiral of every order to the periphery of the correſpondent circle, *viz.* if the number of the order be doubled for the periphery of the circle, and the next antecedent odd number be taken for the ſpiral line; ſo what we have hitherto demonſtrated ſupplys alſo another rule, to define the proportion of the ſpiral ſpace in any order to its circle. For ſince the circles are in a progreſſion of Squares 1, 4, 9, 16, *&c.* but the firſt circle is to the firſt ſpace as 3 to 1 (*i. e.* a, 1 to $\frac{1}{3}$) by *Prop.* 17. and the ſecond to the ſecond as 12 to 7 (*i. e.* as 4 to $2\frac{1}{3}$) by *Prop.* 18. the third to the third as 27 to 19 (*i. e.* as 9 to $6\frac{1}{3}$) by *Schol.* 1. of this. And contemplating both theſe ſeries one by another,

Of the circles, 1, 4, 9.
Of the ſpaces, $\frac{1}{3}$, $2\frac{1}{3}$, $6\frac{1}{3}$.

We ſee the numbers of the ſpaces are produced, if from the ſpuare numbers of the circles you ſubſtract their roots, and add to the remainder $\frac{1}{3}$. Wherefore, if, *e. g.* we were to determine the proportion of the fourth circle to the fourth ſpiral ſpace; the ſquare of 4 *viz.* 16 would give the circle; hence ſubſtracting the root 4, there wiill remain 12, and adding $\frac{1}{3}$ you would have the fourth ſpiral ſpace $12\frac{1}{3}$; and in like manner the ſpiral ſpace $20\frac{1}{3}$ would anſwer to the circle 25, *&c.* And that this is certain is hence evident, that if we multiply theſe numbers 16 and $12\frac{1}{3}$, alſo 25 and $20\frac{1}{3}$ by 3, that we may have thoſe proportions in whole numbers, 48 and 37, 75 and 61, theſe are thoſe very numbers *Archimedes* had hinted at in the *Coroll.* of *Prop.* 25.

CONSECTARY X.

NAY what we have now ſaid, is that very *Coroll.* comprehending alſo that 25*th. Propoſition, viz.* that a ſpiral ſpace of any order is to its correſpondent circle, as the rectangle of the ſemidiameters of this and the preceding circle together with a third part of the ſquare of the difference between both ſemi-diameters to the ſquare of the greateſt ſemidiameter. For, if *e.g.* the proportion of the third ſpiral ſpace to the third circle be required, ſince the ſemidiameter of this third circle is as 3, and the ſemidiameter of the ſecond precedent one is 2, and ſo the difference 1; the rectangle of 2 into 3 *i. e.* 6, together with $\frac{1}{3}$ of the ſquare of the difference will define the third ſpiral ſpace $6\frac{1}{3}$; ſince the third circle may be defined by the ſquare of the ſemidiameter of the greater, *viz.* by 9, and ſo in the reſt; as the numbers we have found ſhew, or further that may be found according to given Rules which may be here ſeen in the following Table.

Orders.	I	II	III	IV	V	VI	VII	VIII	XI	X
Circles.	1	4	9	16	25	36	49	64	81	100
The whole ſpaces, the preced. ones being included.	$\frac{1}{3}$	$2\frac{1}{3}$	$6\frac{1}{3}$	$12\frac{1}{3}$	$20\frac{1}{3}$	$30\frac{1}{3}$	$42\frac{1}{3}$	$56\frac{1}{3}$	$72\frac{1}{3}$	$90\frac{1}{3}$
Separate ſpaces the preced. ones being excluded.	$\frac{1}{3}$	2	4	6	8	10	12	14	16	18

CONSECTARY XI.

OUT of which table it is obvious to ſight, that the ſecond ſpace excluding the firſt is ſextuple of the firſt, as we have already deduced in *Conſect.* 2. *Prop.* 18. and the third ſeparate ſpace double of the ſecond, and the fourth triple of the ſame ſecond, and the fifth quadruple, and ſo onwards.

SCHOLIUM III.

AND this ſhall ſuffice for ſpirals, which comprehends not only the chief Theorems of *Archimedes* of ſpiral ſpaces, but alſo the chief of ſpiral lines (whereof *Archimedes* has left nothing.) If any ſhould have a mind to carry on our method further, he may eaſily demonſtrate after the ſame way what remains in *Archimedes*, and what Dr. *Wallis* in his Arithmetick of Infinites from *Prop.* 5. to the 38, and what others have done on this Argument.

CHAP. V.

Of the Conchoid, Ciſſoid, Cycloid, Quadratrix, &c.

Propoſition XIX.

THE *firſt conchoid of* Nicomedes *Bbb (Fig.* 110.) *on both ſides of the perpendicular cDb approaches nearer always to the directrix or horizontal line AE, and yet will never coincide with it, altho' it be conceived to be produced on both ſides* ad infinitum.

Demonſtration.

For ſince only D*b* is perpendicular to AE, and all the reſt *ab* are ſo much the more inclined to it by how much the more remote they are from the middle one D*b*, and all in the mean while are equal both to it and to one another, by *Def.* 13. it is evident that the points *b* and B will come ſo much the nearer to AE, by how much the farther they recede from the middle line D*b*. And yet becauſe the lines BAC and *bac* are all right ones, whoſe points A, *a*, are in a right line AE it is equally as impoſſible that the point *b* or B, which is always in the conchoid ſhould ever touch this right line, as it is impoſſible that the point C ſhould be in it, by vertue of the aforecited *Def.* Q. E. D.

Propo-

Propoſition XX.

YET no other right line can be drawn between the directrix AE and the conchoid, but what will cut it if produced.

Demonſtration.

For if ſuch a right line be made parallel to AE, as GH, and you make, as DI to IC ſo D*b* to a fourth, which will be greater than IC, as D*b* is greater than DI, and conſequently, if making that an interval you draw the circular arch from C, it will neceſſarily cut the line GH *e. g.* in G. Drawing therefore C*a*G. you'l have as DI to IC ſo *a*G to GC, *i. e.* to that fourth proportional before found, by vertue of *Prop.* 34. *lib.* 1. but as DI to IC, ſo was alſo D*b* to the ſame fourth by Conſtruc. Therefore *a*G and D*b*, which have both the ſame proportion to the ſame quantity, are equal; and conſequently the point G is in the conchoid by virtue of *Def.* 13. and conſequently the right line GH being produced will cut that produced alſo, on both ſides, by the ſame reaſon. Much more will it cut it on either ſide if it be not parallel to the directrix AE, which is very obvious. Therefore no right line can be drawn between the conchoid, *&c.* Q. E. D.

CONSECTARY.

HEnce, beſides orher Problems, that may be very eaſily ſolved, which requires, having any rectilinear angle given ABC (*Fig* 143.) and a point without it, from that point to draw a right line DEF, ſo that part of it EF, which is intercepted between the legs of the angle, ſhall be equal to a given line Z. For if you draw the perpendicular DGH from the given point D through the neareſt leg of the angle BC, and make GH equal to the given line Z, and from the center C at the interval GH deſcribe the conchoid IHK, which will be neceſſarily cut by the other leg of the angle by vertue of the preſent *Prop.* *e. g.* in F, the line DF being drawn will give the intercepted part = GH by the nature of the conchoid, and conſequently = to the given line Z.

SCHOLIUM.

BY means of this Conſectary *Nicomedes* ſolves that noble Problem of finding two mean proportionals, after this way, which we will here ſhew from *Eutocius*, but drawn into a compendium, and ſomewhat changed as to the order. Let two given lines AB and BC (*Fig.* 144.) between which you are to find two mean proportionals, be joined together at right angles, and divide both into two parts in D and E, and having compleated the rectangle ABCL, from L thro' D draw LG to BC prolonged; ſo that after this way GB may become = AL or BC: Having let fall a perpendicular from E cut off from C at the interval CF = AD the part EF, and having drawn FG make CH parallel to it; and laſtly thro' the legs of the angle KCH draw the right line FHK, ſo that the part HK ſhall be equal to the line CF, by the preceding Conſect. and alſo draw the right line KM from K thro' L to the continued line BA: All which being done, CK and AM will be two mean proportionals between AB and BC; which after our way we thus demonſtrate: By reaſon of the ſimilitude of the △ △ MAL and LCK

MA is to LC or AB as AL or BC to CK
b —— *eb* —— *c* —— *ec*

and moreover,

as MA to AD ſo GC to CK *i. e.* FH to HK
b —— $\frac{1}{2}eb$ —— $2c$ —— ec by reaſon of GF and CH being parallel, by *Conſect.* 4. *Prop.* 34. *lib.* 1. therefore ſince HK is = AD = $\frac{1}{2}eb$, FH will be = A = *b*, and conſequently MD = FK, *viz.* both *b* † $\frac{1}{2}eb$, and the ſquare of both = *bb* † *eb* † $\frac{1}{4}eebb$ = □ EF † EK by vertue of the *Pythag. Theor.* Now if to theſe equal quantities you add the equal □ □ DX and EC = $\frac{1}{4}cc$, their ſum, *viz.* □ MD † □ DX *i. e.* □ MX will be *bb* † *ebb* † $\frac{1}{4}eebb$ † $\frac{1}{4}cc$, equal to the ſum of theſe, *viz.* □ EF † □ EC *i. e.* □ CF (by the *Pythag. Theor.* or EX by Conſtruct.) † □ KX; whence theſe two things now follow: 1. That the lines MX and KX are equal. 2. If from thoſe equal ſums you take away the common quan-

tities $\frac{1}{4}eebb + \frac{1}{4}cc$, the remainders will be equal, *viz.* $bb + ebb \propto ecc + eecc$; and (ſince the part taken away, *viz.* bb is manifeſtly to the other part taken away, *viz.* ecc as the remainder ebb to the remainder $eecc$, and the whole with the parts taken away and the remainders are in the ſame proportion by *Prop.* 26. *lib.* 1.) ſeparately alſo bb will $\propto ecc$ and $ebb \propto eecc$. But from the latter equation it follows that

as eb to ec ſo ec to b by vertue of the 19. *Prop. lib.* 1.
AB to CK ſo CK to MA

and by the ſame reaſon it follows from the former Equation

as ec to b ſo b to c
CK to MA ſo MA to BC *i. e.* CK and MA are two mean Proportionals between AB and BC. Q. E. D.

From which deduction you have alſo manifeſt the foundation of that mechanical way, which *Hiero Alexandrinus* makes uſe of in *Eutocius*, *lib.* 2. *of the Sphere and Cylinder*, and which *Swenterus* has put into his practical Geometry *lib.*1. *Tract.* 1. *Prop.* 23. when, *viz.* having joined in the form of a rectangle the given right lines AB and BC (*Fig.* 145.) and continued them at the other ends, he ſo long moves the ruler in L, having a moveable center, backwards and forwards, 'till XK and XM by help of a pair of compaſſes are found equal. To which, another way of *Philo*'s is not unlike, and flows from the ſame fountain, wherein, having made on AC a ſemicircle, the moveable ruler in L is ſo long moved backwarks and forwards, until LM and NK are found equal: Which ſeems to *Eutocius* to be more accommodated to practice, and eaſier to be perform'd by help of a ruler divided into ſmall equal particles.

Propoſition XXI.

If from any point of the other diameter in the generating circle e.g. from G (*Fig.* 111. *n.* 1.) *you draw a perpendicular* GE *thro' the ciſſoid of* Diocles. *the lines* CG, GE, GD, *and* GH *will be continual proportionals.*

Demonſtration.

For ſince GE and IF, as right ſines, and alſo GD and IC as verſed ſines of equal arches by the Hypoth. are equal; you'l have as ID to IF (*i. e.* CG to GE) ſo IF to IC (*i. e.* GE to GD) *per n.* 3. *Schol.* 2. *Prop.* 34. *lib.* 1. But GD is to GH as ID to IF (*i. e.* as GE to GD) by the forecited *Prop.* 34. *lib.* 1. Therefore CG to GE, GE to GD, and GD to GH, are all in the ſame continual proportion. Q. E. D.

CONSECTARY.

HEnce it was eaſie for *Diocles* to find two mean proportionals *x* and *y* between two given right lines V and Z; (*Fig.* 146) for he made (having firſt deſcribed his curve DHB) as V to Z ſo CL to LK, and having drawn CKH to the curve, and thro' H the perpendicular GE, he had between CG and GH two mean proportionals GE and GD by vertue of the preſent *Prop.* when in the mean while CG the firſt would be to GH the laſt, as CL to LK, *i. e.* as the firſt V to the laſt Z given by vertue of the Conſtr. Therefore nothing remain'd but to make, 1. as CG to GE ſo V to X; and laſtly, as GE to GD ſo *x* to *y*.

SCHOLIUM.

IT may not be amiſs to mention here another way of finding two mean proportionals between any two given lines by the help of two Parabola's, which *Menechmus* formerly made uſe of, *viz.* by joining at right angles the given lines AB and BC (*Fig.* 147) and prolonging them as occaſion ſhall require thro' E and D; and then deſcribing a Parabola about BE as its axis, ſo made that BC ſhall be its *Latus Rectum*, and in like manner deſcribing another Parabola about BD as its ax, that ſhall have AB for its *Latus Rectum*, and that ſhall cut the former in F: Which being done, the ſemiordinate FE (or BD which is equal to it) being drawn to the point of Interſection F, will be the two mean proportionals ſought. For by vertue of the fourth Conſectary of *Prop.* 1. of this Book, DF or

BE is a mean proportional between AB and BD, and in like manner EF or BD is a mean proportional between BE and BC, and conſequently as AB to BE ſo BE to BD, and as BE to BD ſo BD to BC; Q. E. D.

To this way of *Menechmus* that of *Des Cartes* is not unlike, which he gives us *p. m.* 91. except only that inſtead of two Parabola's, he makes uſe only of one and a circle in room of the other: In imitation of whom *Renatus Franciſcus Sluſius* has ſince ſhewn infinite methods of doing the ſame thing by help of a circle, and either infinite Ellipſe's or Hyperbola's, in his ingenious Treatiſe which he thence names his *Meſolabium.*

Propoſition XXII.

ANY ſemiordinate of the Cycloid as BF (Fig. 148.*) or bf is equal to its correſponding Sine in the generating circle as BD, bd, together with the arch of that ſine AD or Ad.*

Demonſtration.

For the motion of the point A deſcribing the ſemi-cycloid AFE, by *Def.* 11. is compounded of the motion of the orb (or wheel) B along the ſemi-circle ADC, and of the motion of the centre along the right line BC eqnal to CE, and conſequently to the ſemi circle it ſelf, or motion of the orb, Therefore as the point A moving to E by the motion of the orb (or wheel) moved or was carried from the diameter AC thro' the whole ſemi circle ADC 'till it came to AC again, and by the motion of the Centre paſſes thro' the whole ſpace BG or CE, which is equal to the ſemicircular arch; thus the ſame A, when come to F, will have deſcrib'd the quadrant AD, by moving from the diameter AC the quantity of the ſine BD, and moreover by the motion of its centre (which is equal to the motion of the Orb) moves from AC the ſpace of DF: And ſo the ſemiordinate BF will be equal to the arch AD and to its ſine BD taken together; and in like manner the ſemiordinate *bf* will be equal to the arch A*d* and its ſine *bd*, &c. Q. E. D.

CONSECTARY I.

HEnce may be eaſily aſſign'd by help of the cycloid a right line equal to the ſemi-periphery or any given arch AD or A*d*; *viz.* CE double, or taken twice for the whole circumference, and ſingle for the ſemi-periphery or half circumference, DF for the quadrant A*d*, *df* for the arch AD, *&c.*

CONSECTARY II.

WHerefore the quadrature of the circle may be geometrically obtain'd according to *Conſect.* 2. of *Def.* 15. *lib.* 1.

CONSECTARY III.

IF you take B*e*, *be*, double of the ſines BD, *bd*, *&c.* ſo that all the indiviſibles *bd* taken together, may be to all the indiviſibles *be* taken together as BD to B*e*, a curve deſcribed thro' the points *e* will be an ellipſis by *Prop.* 11. and the curvilinear ſpace ADC*e*A will be equal to the ſemicircle ACDA.

CONSECTARY IV.

AND ſince DF (*i.e.* D*e* † *e*F) is equal to the quadrant DA, by vertue of the preſent Prop. BD † FG will be alſo equal to the quadrant (becauſe the whole BG or CE is $=$ to a ſemicircle) and conſequently *e*F and FG will be equal; in like manner ſince *df* both above and below is equal to the arch *d*A, below *bd* † *fg* will be $=$ to the remaining Arch *d*C: and above *bd* † *ef* (*i. e. df*) will be equal to the equal arch *d*A. Therefore *ef* above and *fg* below are equal, and (ſince the ſame may be ſhewn of all the indiviſibles of the ſame ſort throughout) the trilinear figure FGE will be equal to the trilinear *e*FA.

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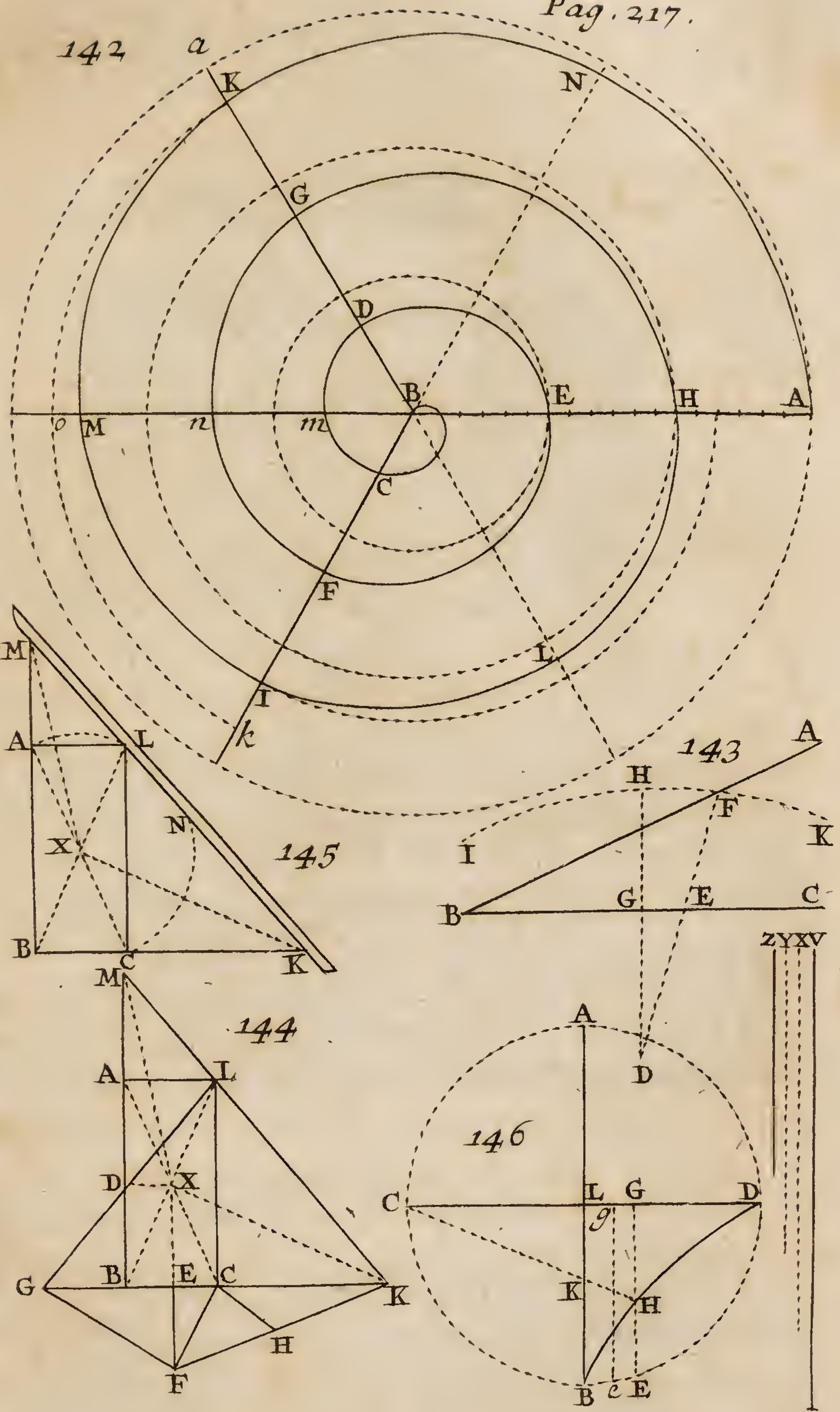
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Propoſition XXIII.

THE cycloidal ſpace is triple of the generating circle i. e. *the ſemi-cycloidal ſpace AECA is triple of the ſemi-circle ADCA.*

Demonſtration.

Since the parallelogram BCEG is equal to the whole circle by *Conſect.* 2. of *Def.* 15. *lib.* 1. *i e.* to the ſemi-ellipſe A*e*CA, by the preſent conſtruction the Trapezium C*e*GE will be equal to the quadrant of the ellipſe or the ſemi-circle. But the trilinear ſpace FEG is = to the trilinear ſpace FA*e*, by the fourth *Conſect.* of the preced. therefore alſo the trilinear ſpace A*e*CEA is equal to the ſemi-circle. Therefore the whole cycloidal ſpace is equal to the three ſemi-circles. Q. E. D.

Or thus.

Since the whole parallelogram AE is equal to two circles and the ſemi-ellipſis A*e*CA to one: the remaining ſpace A*e*CEC to one circle, and its half A*e*GC to a ſemi-circle. But the trilinear ſpace A*e*F is equal to the trilinear ſpace FGE by *Conſect.* 4. of the preced. Therefore the one being ſubſtituted in the other's place the trilinear ſpace AFEC will be equal to the ſemi-circle: Therefore the remainder of the Parallelogram, *i. e.* the cycloidal ſpace AFECA will be equal to three ſemi-circles. Q. E. D.

SCHOLIUM.

TO theſe ſhort Demonſtrations, which we confeſs we owe for the moſt part to *Hon. Faber*, we will ſubjoin another ſomewhat more prolix, but yet not unpleaſant, which we find in *Carolus Renaldinus, lib.* 1. *de Reſol. & Compoſ. Math.* p. 299. But here we will give it the Reader more plain, and free from all Scruples, and likewiſe much eaſier. It is perform'd in theſe inferences, 1. That the right-lined Parallelogram A*ba*B (*Fig.* 149 *n.* 1) is equal to the curvilinear ſpace A*bda*BDA. 2.

2. That as that is divided into two equal parts by its right lined diagonal A*a*, ſo likewiſe is this by the ſemi-cycloid A*aa*, ſo that the right lined triangle A*a*B is equal to the curvilinear ſpace A*aa*BDA. 3. Therefore the one as well as the other is equal to the generating circle; and conſequently, 4. if to this curvilinear ſpace there be added the ſemi-circle ADBA the ſemi-cycloidal ſpace A*aa*BA will be equal to three ſemi-circles. The firſt is evident, while if you take from the right lined parallelogram the ſemi-circle ADBA on the one ſide, and on the other add the ſemi-circle *abda*, there will ariſe the curvilinear Parallelogram. The third is evident from *Conſect.* 2. *Def.* 15. *lib.* 1. Becauſe the line B*a* is equal to the ſemi-periphery, which multiplyed by the diameter BC gives the area of the circle. The fourth is ſelf evident; and ſo there remains only the ſecond to be demonſtrated, *viz.* That the curvilinear parallelogram is divided into two equal parts by the cycloid, *i. e.* that the external trilinear Figure A*adb*A is eqaal to the internal one A*aa*BDA; which may be thus ſhewn: Having divided the baſe B*a* into three equal parts, and drawn thro' them three ſemi-circles, and moreover the tranſverſe right lines D*d* and E*e* thro' the interſections of the ſem-icircles and the cycloid; it is certain from the geneſis of the cycloid, by vertue of the Conſ. of *Def.* 11. that as the right line *a*1 is a third part of the whole *a*B, ſo the arch 1*a* is a third part of the generating periphery, and by the ſame reaſon the arch 2*a* two thirds, and ſo the remaining arch *a*11 alſo $\frac{1}{3}$; inſomuch that the firſt arch 1*a*, and the laſt *a*11, and conſequently their right ſines *af*, *ag*, and likewiſe their verſed ones *f*1, *g*11, are equal, and ſo the curvilined partial Parallelograms, both above and below, all upon equal baſes, and of the ſame height (*viz.* the two linear ones *ae*21 and *da* 11. 1.) are equal among themſelves, and ſo likewiſe the two pricked or pointed ones D*a*2B and *ae b*1. Wherefore if now the baſe B*a* (*n.* 2) be conceived to be divided into ſix equal parts, and having drawn ſemi-circles, and tranſverſe lines thro' their interſections with the cycloid, the arches

$$\left.\begin{matrix} 1a \\ \text{and } Va \end{matrix}\right\}\text{will be } \frac{1}{6}\quad \left.\begin{matrix} 2a \text{ and} \\ 1Va \end{matrix}\right\}\frac{1}{3},\quad \left.\begin{matrix} 3a \text{ and} \\ IIIa \end{matrix}\right\}\frac{1}{2} \text{ of a ſemi-circle,}$$

and ſo the verſed ſines of each, *i. e.* the altitudes of the correſponding parallelograms will be equal, and conſequently the parallelograms of the internal and external trilinear ſpace that are alike noted or ſigned will be equal to each other. Now this inſcription of curvilinear parallelograms always reſpectively equal both in number and magnitude, ſince it may be continued in both the trilinear figures *ad infinitum* ; it will evidently follow, that the trilinear figures themſelves, whoſe infinite inſcripts are always equal, will be likewiſe equal to one another.

Propoſition XXIV.

THE baſe of the quadratrix AE (Fig. 150) and the ſemidiameter of the generating quadrant AD and the quadrant it ſelf BD are in continual Proportion.

Demonſtration.

For the quadrant DB is to the radius DA as the arch IB to the perpendicular H*e* by *Conſect.* 1. *Def.* 16 and I*b* is to H*e* as A*b* to A*e* by *Prop.* 34. *lib.* 1. But the arch IB (if it be conceiv'd to be leſs and leſs *ad infinitum*) will at length coincide with I*b*, as ending in the ſame moment in the point B, wherein H*e* will end in the point E, and ſo A*e* will end in AE and A*b* in AB. Therefore at length DB will be to DA as IB (*i. e.* I*b*) to H*e*, *i. e*, as A*b* to A*e*, *i. e.* as AB (or DA) to AE. Q. E. D.

SCHOLIUM I.

CLavius about the end of the ſixth Book of *Euclid*, and others, demonſtrate this indirectly by a deduction *ad Abſurdum*, or concluding the oppoſite much after this manner: If DA or AB is not to AE as DB to DA, ſuppoſe it to be ſo to the greater A*f* or the leſs A*e*. In the firſt caſe therefore, becauſe AB is to A*f* as DB to DA *per Hypoth. i. e* as K*f* to A*f* the quadrant K*f* and the radius AB or DA will be equal. But as BD is to IB ſo is K*f* to H*f* by reaſon of the ſimilitude of the arches; and as BD to IB ſo alſo DA (or = K*f*) to the ſine H*e*, by

Conſect. 1. *Def.* 16. Therefore the ſine H*e* and the arch H*f* (to which the ſame K*f* bears the ſame proportion) will be equal; which is abſurd. In the latter caſe, becauſe AB would be to A*e* as IB to DA by the hypoth. *i. e.* as L*e* to A*e*, the quadrant L*e* and the radius AB or DA would be again equal. But as BD is to IB ſo is L*e* to M*e* by reaſon of the ſimilitude of the arches; and as BD to IB ſo alſo is DA (*i. e.* = L*e*) to H*e* by *Conſect.* 1. *Def.* 16. Therefore the tangent H*e* and the arch M*e* (to which the ſame L*e* bears the ſame proportion) will be equal, which is again abſurd. Wherefore BD is to DA, not as DA to a greater A*f* or a leſs A*e*; therefore as DA to AE. Q. E. D.

CONSECTARY I.

WHerefore it is evident from what we have deduced, if by means of the baſe of the quadratrix AE you draw a quadrant the ſide of the quadratrix DA will be equal to it, and conſequently double of the ſemi-periphery, and quadruple of the whole periphery.

CONSECTARY II.

IT is evident alſo that you may obtain a right line equal to the quadrant DB of any given circle, if, having deſcribed a quadratrix, you make as AE to AD ſo AD to a third equal to the quadrant DB: Which third proportional taken four times will be equal to the whole periphery.

CONSECTARY III.

YOU may alſo obtain a right line equal to any leſs arch, if you make, as DA to H*e* ſo a third proportional found (*i. e.* the quadrant DB) to a fourth, by vertue of *Conſ.* 1. *Def.* 16.

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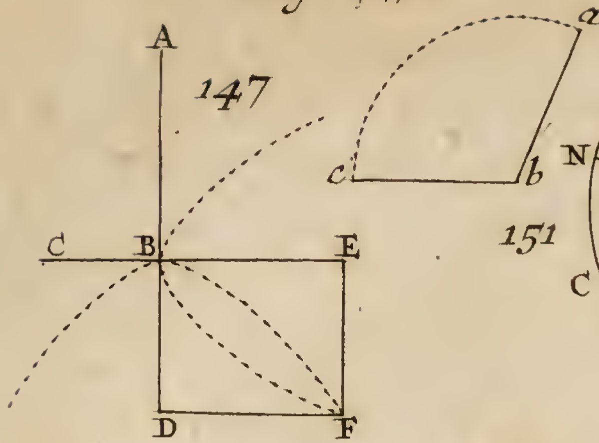
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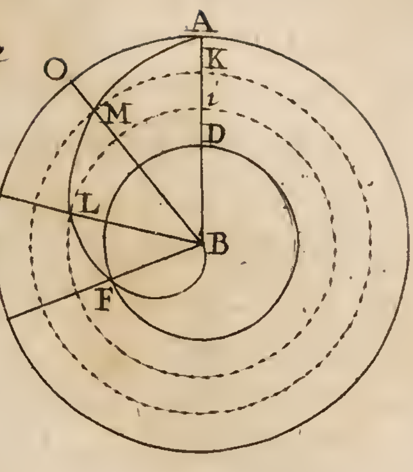
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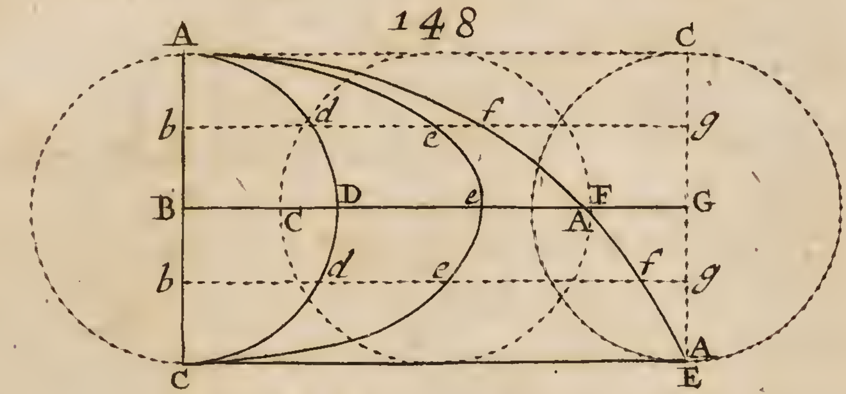
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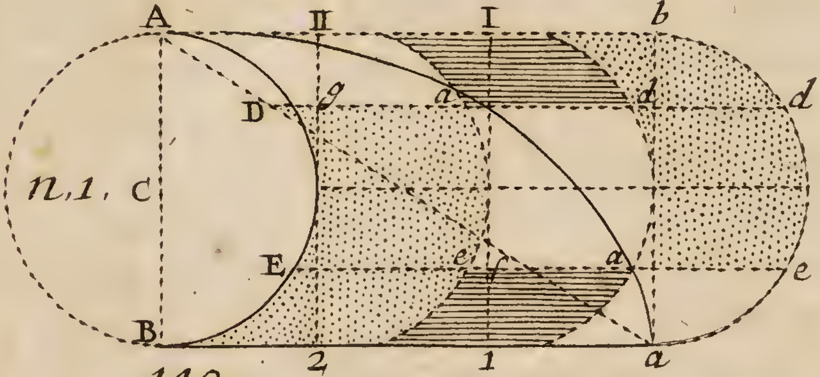

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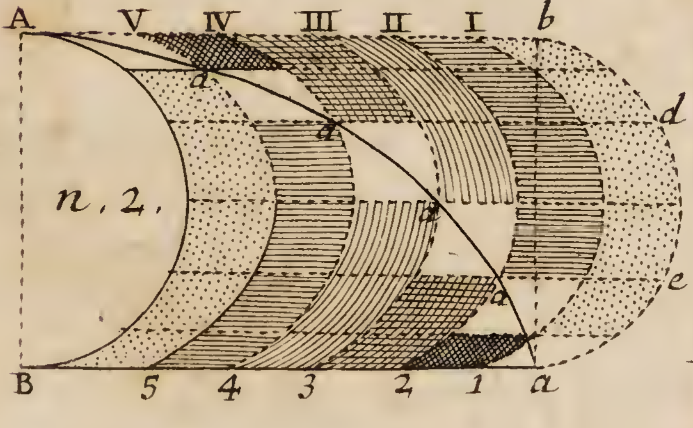

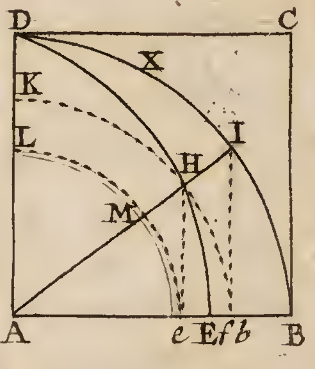
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CONSECTARY IV.

THE quadrature of the circle therefore, by vertue of *Conſ.* 2. *Def.* 15. *lib.* 1. as likewiſe the triſection of an angle, by vertue of *Conſect.* 2. *Def.* 16 *lib* 2. may be Geometrically obtain'd if the Quadratrix might be number'd among Geometrical Curves.

SHCOLIVM II.

C*Lavius* was alſo of this Opinion in the book afore mentioned, who thought that if the quadratrix be excluded out of the number of geometrical curves, by the ſame reaſon you may alſo exclude the ellipſe, parabola, and hyperbola, ſince they as well as this are commonly deſcribed thro innumerable points. But by that great Man's leave, we may deny this conſequence, by the ſame reaſon as *Des Cartes* has deny'd the converſe of it in his *Geom* p. 18. and 19. by vertue of which he ſuſpects the ancients took the conick ſections, *&c.* for mechanick or non-geometrick lines, becauſe they did the *ſpiral*, quadratrix, *&c.* for ſuch. But this is the difference between the deſcription of the quadratrix and the conick ſections thro' points, that all and every of the points of the conick ſections, relating to any given point of the axis, may be geometrically determin'd; but all the points of the quadratrix promiſcuouſly related to any point of the generating quadrant, cannot be geometrically determin'd, but only thoſe which reſpect ſome certain point, from which the quadrant may be divided into two arches of known proportion. For if, *e g.* in the quadrant BD the point X be given at pleaſure, it will be impoſſible by *Clavius*'s Rule to define a point of the quadratrix anſwering to it, becauſe the proportion of the arches DX and BX is unknown,. and conſequently neither can a proportional ſection of the right line AD be made: Not to mention that the laſt point E (which is the primary and moſt neceſſary one to the quadrature) even by *Clavius*'s own confeſſion cannot be geometrically defined. We may paſs the like judgment on *Archimedes*'s ſpiral and ſuch like curves, which are conceiv'd to be deſcribed by two motions independent on one another; as will be manifeſt to any one who compares the geneſis of the ſpiral with that of the quadratrix and what we have hitherto ſaid. Whence neither will *Monantholius*'s triſection of a given angle (which he eſſays) by means of a ſpiral be enough geometrical; which in his Book *de Puncto*, *Cap.* 7. p. 24. he attempts to perform thus: To the centre of a deſcribed ſpiral and its firſt helical or ſpiral line BA (*Fig.* 151.) he applies the angle ABC equal to the given one *abc*; then having drawn circles thro' F and A where the legs of the angle cut the ſpiral, he divides the intermediate ſpace DA into three equal parts in I, and K: And then thro' theſe points he draws circles cutting the helix in L and M; and laſtly having drawn BLN, BMO, he eaſily demonſtrates from the geneſis of the ſpiral that the arches AO, ON, NC are equal. And ſo after the ſame manner not only any angle or arch, but the whole periphery may be geometrically divided into as many parts as you pleaſe; only ſuppoſing that this ſpiral line may be numbred among geometrical ones; as we have heretofore hinted that the cycloid, conchoid, ciſſoid, and logarithmical curve, *&c.* might be; and we have above ſixteen Years ago declared our opinion for it in our *German* Edition of *Archimedes*; and now are therein confirm d by thoſe celebrated Mathematicians *Leibnitz*, *Craige*, *&c.* who number lines of this kind, altho' they cannot be expreſſed by our common equations, among geometrical ones, notwithſtanding the contrary opinion of *Des Cartes*, *&c.* becauſe they admit of equations of an indefinite or tranſcendent degree, and are capable of a *Calculus* as well as others, tho' it be of a nature and kind different from that commonly uſed. See the *Acta Erud. Lipſ. ann.* 84. p. 234. and *ann.* 86. p. 292. and 294.

CHAP. VI.

The Concluſion, or Epilogue of the whole Work.

NOW we may at length underſtand what *Honoratus Fabri* delivers concerning the diſtribution of figurate magnitudes into certain Claſſes, in his *Synopſis Geom.* p. 57. and the following.

1. The firſt Claſſe contains elementary figures, or equal indiviſibles, ſuch as, 1. All Parallelograms, as the Square, Oblong, Rhombus and Rhomboid, the elements whereof are equal right lines, as in *Def.* 12. *lib.* 1. 2. Convex or concave Surfaces, the elements whereof are curve lines moved thro' right lines by a parallel motion; among which are chiefly reckon'd cylindrical ſurfaces, whereof ſee *Def.* 16. *lib.* 1. about the end. 3. Parallelepipeds, and among them the cube, whoſe indiviſibles are ſquares, or other Parallelograms. 4. Priſms made by the motion of a Triangle, Trapezium, or any Polygonous Body, along a right line, all the indiviſibles whereof are conſequently ſimilar and equal to the generating plane.

2. The ſecond Claſſe contains Figures whoſe Elements decreaſe in a ſimple arithmetical Progreſſion; ſuch are, 1. Triangles, as is evident from *Prop.* 37. *lib.* 1. 2. The circle, and its Sectors, as reſolvible into concentrick Peripheries according to *Conſ.* 1. and 3. of the aforecited *Prop.* 3. The Cylinder as reſolvible into concentrick cylindrick Surfaces, as its indiviſibles. 4. The Surface of a Cone, whoſe elements are circular Peripheries, and alſo of the Pyramid whoſe indiviſibles are ſimilar angular Peripheries every where increaſing in arithmetical Progreſſion. 5. The *Parabolick Conoid*, whoſe indiviſibles are Circles according to the proportion of the abſciſſa's in arithmetical Progreſſion, by vertue of *Prop.* 14. *lib.* 2. *&c.*

3. The third Claſſe contains elementary Figures increaſing in duplicate arithmetical Progreſſion; ſuch are, 1. The Pyramid and Cone, the firſt whereof may be reſolv'd into angular Planes, the ſecond into circular ones increaſing according to a ſeries of ſquare numbers; as is evident from *Prop.* 38.

lib. 1. and its Conſectary. 2 The trilinear parabolick Space, as defin'd *Prop.* 10 *lib.* 2. by the letters E*b* HK. 3. The Sphere, as far as it may be reſolved into ſpherical concentrick Surfaces, every one whereof may be conſider'd as a baſe, taking the ſemidiameter for the altitude. 4. The Cone, as reſolvible into parallel conical Surfaces deſcrib'd by the parallel indiviſibles of the Triangle. 5. The remainder of a Cylinder after an Hemiſphere of the ſame baſe and altitude is taken out, according to *Schol.* 1. of *Prop.* 39. *lib* 1.

4 The fourth Claſſe would comprehend all magnitudes reſolvible into elements or indiviſibles increaſing in triplicate, quadruplicate, *&c.* Arithmetical Progreſſion; ſuch we have not treated of, but may be found among Planes terminated by Curves of ſuperior Genders; ſee *Fabri*'s Synopſis, p. m. 67.

5. The fifth Claſſe is of thoſe Magnitudes, whoſe indiviſibles decreaſe, proceeding from a ſquare number by odd numbers, as, 36, 35, 32, 27, 20, 11, *&c.* ſuch are, firſt, an *Hemiſphere*, as is evident from *Prop.* 39. *lib.* 1. 2. An *Hemiſpheroid*, as in *Prop* 15 *lib* 2. 3. A *Semi-parabola*, as may be gather'd from the demonſtration of *Prop.* 10 *lib.* 2. For ſince the indiviſibles of the circumſcrib'd trilinear figure *eb* are found in a duplicate arithmetical Progreſſion, $\frac{1}{4}$, $\frac{4}{4}$, $\frac{9}{4}$, $\frac{16}{4}$, the indiviſibles of the ſemi-parabola will neceſſarily be $\frac{16}{4}$, $\frac{15}{4}$, $\frac{12}{4}$, $\frac{7}{4}$, *&c.*

6. We may make a ſixth Claſſe of thoſe Magnitudes whoſe indiviſibles decreaſe in a like Progreſſion, not of the numbers themſelves deſcending by odd ſteps from a given ſquare, but of their roots, which are for the moſt part ſurd ones; ſuch as is firſt, the *Semi circle*, as is evident from *Prop.* 43 *lib.* 1. and by vertue of *Prop.* 5. *lib.* 2. and alſo the ſemi-ellipſe, *&c.*

7. The ſeventh Claſſe comprehends thoſe Magnitudes, whoſe Elements are in a Progreſſion of a double ſeries of numbers, as in the *Parabolick Conoid*, as may be ſeen in the *Scholium* of *Prop.* 16. *lib.* 2.

But, to omit the other Claſſes of Magnitudes of a ſuperiour Gender, the conſideration whereof theſe Elements either have not touch'd on, or only by the by; (which any one who pleaſes may ſee in *Faber*'s Synopſis, eſpecially thoſe which he comprehends under the ſixth and ſeventh Claſſes, p. 70. and the following) about thoſe we have here particularly noted, there remain only two things to be taken notice of. 1. That ſince in the firſt Claſſe we place Parallelograms and Cylinders, in the ſecond Triangles, in the third Pyramids and Cones, in the fifth Hemiſpheres, in the ſixth ſemi-circles, *&c* We may with *Hon. Faber* call the firſt Claſſe, that of Cylindrical or Parallelogrammatick Figures; the ſecond, the Claſſe of Triangular Figures; the third, of Pyramidals; the fifth, of Hemiſpherical Figures; the ſixth of ſemicircular ones, *&c.* 2. That having ranged or reduced after this manner homogeneous Figures, or thoſe of like condition, to a few Claſſes, their dimenſion, and conſequently almoſt the whole buſineſs of meaſuring may be very compendiouſly reduc'd to a few Rules; whereof we will here give the Reader a ſhort Specimen, in the following

CONSECTARYS.

I. THE dimenſion of Parallelogrammatick Figures, *i. e.* of thoſe of the firſt Claſſe, may be had, by multiplying the whole baſe by the whole altitude: See *lib.* 1. *Def.* 12. *Conſ.* 7. *Def.* 18. *Conſ.* 6. *Def.* 16. *Conſ.* 3. and 4.

II. The dimenſion of Triangular Figures, *i. e.* of thoſe of the ſecond Claſſe, may be had by the multiplication of the whole Baſe by half the Altitude, or of half the Baſe by the whole altitude; [ſee *lib.* 1. *Def.* 12. *Conſect.* 8. *Def.* 15. *Conſect.* 2. *Def* 18 *Conſect.* 4 *lib.* 2. *Prop.* 14.] and their Proportion is to their reſpective circumſcribing Parallelograms, as 1 to 2; [ſee beſides the *Prop.* already cited, *lib.* 1. *Prop.* 37. and its firſt *Conſect.*]

III. The dimenſion of Pyramidals, *i. e.* of magnitudes of the third Claſſis, may be obtain'd by the multiplication of the Baſe by the third part of the of Altitude; [ſee *lib.* 1. *Def.* 17. *Conſect* 3. and 4. and *Def.* 20. *Conſect.* 1. &c.] and their Proportion to the correſponding Figures of the Claſſe of the ſame Baſe and Altitude is as 1 to 3. [ſee beſides the *Prop.* already cited *Prop.* 38. *lib.* 1. and its *Conſ.* *Prop.* 39. and *Schol.* 1. *lib.* 2. *Prop.* 10. *&c.*

IV. The Proportion of Hemiſpherical Magnitudes, *i. e.* of the fifth Claſſis to correſponding ones of the firſt Claſſe of the ſame Baſe and Altitude is as 2 to 3; [ſee *lib.* 1. *Prop.* 39. *lib.* 2. *Prop.* 10. and 15.] and ſo their dimenſion may be had by multiplying theit Baſe by $\frac{2}{3}$ of their Altitude.

V. The Proportion of ſemi-circular Magnitudes, *i. e.* of thoſe of the ſixth Claſſe to ſo many correſponding ones of the firſt Claſſe of the ſame Baſe and Altitude cannot be expreſſed by whole numbers or by a ſmall fraction [ſee *lib.* 1. *Prop.* 43. and *lib.* 2. *Prop.* 11.] and conſequently their exact numeral dimenſion cannot be had.

AN

INTRODUCTION

TO

SPECIOUS ANALYSIS,

OR,

The New Geometry, chiefly according to the Method of *Des Cartes*,

But much facilitated by later Inventions, *&c.*

By *J. CHRIST. STURMIUS.*

THE
PREFACE
TO THE
READER.

SINCE *thoſe ingenious Mathematicians of this preſent Age, which is now drawing to a Concluſion,* Vieta, Ougthred, Harriot, Cartes, Schooten, Beaune, Van Hudde, Heuraet, de Witte, *and* Sluſius, *and ſeveral other Famous Men coeval with them, have by their Endeavours improved the* Algebra *of the Ancients, raiſed it to vaſtly an higher pitch, and brought it from Numbers to univerſal Symbols, and not only found the excellent uſes of it in Geometry themſelves, but alſo communicated them to others; almoſt all Countries have furniſhed us with ſome excellent Perſons, who treading in the footſteps of their Predeceſſors, have endeavoured to advance it yet further: And even our Times are not without thoſe of the higheſt rank, as* Wallis, Baker, Renaldinus, Mengolius, Huggens, Malbranch, Leibnitz, Craan, *and ſeveral others, who endeavour to promote this Science, deſervedly reputed the very* Apex *of Humane Reaſon, and carry it more and more towards its utmoſt Perfection by dayly augmenting it with new and curious Inventions. But in the mean time, while theſe ingenious Men wholly buſie themſelves in promoting it, there are few found who condeſcend to explain the firſt Principles of it, and ſhew a ready way to young Beginners to arrive at the knowledge of thoſe Inventions. It is not long ſince a certain Friend of mine, who for ſome time has publickly and ſucceſsfully taught theſe Sciences,* *complain'd to me by his Letters, of the want of a good Guide to Specious Analyſis, whereby he might inſtill the Principles of tha admirable Art into his Auditors. To whoſe Deſires being not juſt then at leiſure to ſatisfie, I immediately after projected this Introduction, which at length you ſee finiſhed, in the Form we now preſent you with it, wherein we have all along conſulted to ſui the Endeavours of young Beginners, as far as poſſible ; as w thought our ſelves engaged by the Duties of our Profeſſorſhip to do & have compriz'd the Precepts of the Art in ſix or ſeven Pages, and accommodated Examples of every kind to illuſtrat them. Wherein if I have but indifferently accompliſh'd my Deſign, I ſhall not think my Labour loſt. We here add it to ou* Matheſis Enucleata, *both as being properly a part of it and more eſpecially, becauſe this Introduction preſuppoſes th Reader to be acquainted with the firſt Principles of Speciou Computation, which we have therein laid down. And ſo w commit our Endeavours to the Peruſal and Cenſure of the Candid Reader.*

INTRO

INTRODUCTION

TO

SPECIOUS ANALYSIS.

THE *Analytick Art*, or *Specious Analyſis*, is ſolely ſubvervient to finding of Theorems, and reſolving Problems, by leading us from certain *Data* or given Quantities, into the knowledge of unknown and ſought ones, by a Chain of certain and infallible Conſequences: This admirable Artifice may be reduced to four Primary Heads, *viz. Denomination*, *Reduction*, *Equation*, and *Effection (if the Problem be a Geometrical one)* or *Conſtruction.*

I. *DENOMINATION.*

BY Denomination is underſtood a preparatory impoſition of Names peculiar to each Quantity, whereby every one of the Quantities given or ſought, are denoted by one or more peculiar Letters of the Alphabet at pleaſure, but with this (arbitrarious) difference, that known or given quantities are mark'd by the former Letters of the Alphabet, *a*, *b*, *c*, *&c.* and the unknown or ſought ones by the latter, *z*, *y*, *x*, *&c.* But although this impoſition of Names, is, as we have ſaid, altogether arbitrarious, yet there often happens not a little facility to the Solution it ſelf, by its being choſen as accommodate as poſſible to the conditions of the quantities given and ſought; which any one will learn better by Uſe than Precepts: As we find that both Theorems may be demonſtrated, and Problems reſolved *e. g.* by an extraordinary Compendium, if we denote any reaſon of two given Homogeneous quantities by *a* and *e a*, *b* and *i b*, *d* and *o d*, *&c.* (*viz.* by expreſſing the Names of the Reaſons by *e*, and *i*, and *o*, *&c.*) and continued proportionality by a, ea, e^2a, e^3a, *&c.* and diſcontinued or diſcrete by $b\ i\ b$, $c\ i\ c$, $d\ i\ d$, or after the like manner, as we have done in our *Math. Enucl. Lib.* 1. *Cap.* 2, 3, 4, 7. and *Lib.* 2. *Cap.* 1. *&c.*

II. *EQUATION.*

HAving thus given each quantity its Name, and making no further diſtinction between the quantities given and thoſe ſought, but treating them all promiſcuouſly, and as already known, you muſt carefully ſearch into and diſcuſs all the Circumſtances of the Queſtion, and making various Compariſons of the quantities, by adding, ſubſtracting, multiplying, and dividing them, *&c.* 'till at length, which is the chief aim and deſign of it, you can expreſs one and the ſame quantity two ways; which is that we call an *Equation*: And you muſt find as many of theſe Equations, or Equalities of literal quantities, (as expreſſing the ſame thing) as there are ſeveral unknown quantities in the Queſtion, independent on each other, and conſequently denominated by ſo many different Letters, z, y, x, *&c.* But if ſo many Equations cannot be found, after having exhauſted all the Circumſtances of the Queſtions by one or two Equations; that is a ſign the other unknown quantities may be aſſumed at pleaſure: Which the Examples we ſhall hereafter bring will more fully ſhew.

But as here alſo (as likewiſe in all this Art) Ingenuity and Uſe do more than Rules and Precepts; yet we will here ſhew the principal Fountains, for the ſake of young Beginners, whence Equations, according to circumſtances obvious in the Queſtion, are uſually had. Theſe are partly Axioms ſelf evident, *E. g.*

That the whole is equal to all its parts taken together.

That thoſe quantities which are equal to one third, are equal among themſelves.

That the Products or Rectangles under the Parts or Segments, are equal to the Product of the whole.

Partly ſome univerſal Theorems that are certain and already demonſtrated, as, *Three (α) continual Proportionals being propoſed, the Rectangle of the Extremes is equal to the Square of the mean.*

(β) Four being propoſed, whether in continued or diſcontinued Proportion, the Product or Rectangle of the Extremes is equal to that of the Means.

And ſeveral others ſuch like, which we have demonſtrated in *Cap.* 2, 3, and 4. *Lib.* 1. of our *Matheſis Enucleat.* partly in the laſt place, ſome particular Geometrical Theorems already demonſtrated, as *e. g.* that common Pythagorick one.

That in rightangled Triangles (γ) the Square of the Hypothenuſa is equal to the two Squares of the ſides.

That the Square of the Tangent of a (δ) Circle is equal to the Rectangle of the Secant and that Segment of it that falls without the Circle; the firſt whereof, we have demonſtrated, *Lib* 1. *Math. Enuc. Def.* 13: *Schol.* and alſo *Prop.* 34. *Conſect.*8. alſo *Prop* 44. after various ways; to which may be numbred *Prop.* 34. with *Schol.* 11. *n.* 3. *Prop.* 37. and following, *Prop.* 45. and 46. alſo the 48. and ſeveral others in *Lib.* 1. *Math. Enucl.* and likewiſe *Lib.* 2 *Prop.* 1, 2, 3, and ſeveral following. And as for Examples both of Denomination, and Equations found after various ways, you may ſee them hereafter follow, and ſome we will here give you by way of Anticipation.

III. *REDUCTION.*

An Equation thus found muſt be reduc'd, *i. e.* thoſe two equal quantities, which for the moſt part are very much compounded of the quantities given and ſought together, muſt be reduc'd to ſuch a form, by adding or ſubſtracting ſomething to or from each part, or multiplying and dividing by the ſame, *&c.* that the unknown or ſought quantity alone, or its Square or Cube or Biquadrate, *&c.* may be found on the one ſide, and on the other the quantity expreſs'd by meer given or known Letters, or affected with the unknown and ſought ones; ſuch are theſe Forms which follow, diſtinguiſh'd by their Names prefix'd to them,

---

(α) *Eucl.* 6. 17.
(β) *id.* 6. 16.
(γ) *id.* 1. 47.
(δ) *id.* 3. 36.

A ſimple Equation, $z = b$, or $y = \frac{ab}{c}$.

A pure Quadratick, $yy = ab$, or $xx = \frac{aa - bb}{2}$.

A pure Cubick, $z^3 = abc$, or $y^3 = \frac{a^3bb}{cd}$.

An affected Quadratick, $x^2 = -ax + b^2$; or,
$y^2 = 2by + \frac{2abc}{d}$.

An affected Cubick, $z^3 = az^2 + b^2z - c^3$ &c.
A Biquadratick, $y^4 = ay^3 + b^2y^2 - c^3y + d^4$ &c.

To one of which, or ſome other like them, when you have reduc'd your Equation firſt found, there are Rules in readineſs, whereby the Value of the unknown or ſought quantity z, or y, or x, may be either expreſſed in Numbers, if the Queſtion be an Arithmetical one, or geometrically determin'd if it be a Geometrical one: Which is that we call *Effection* or *Conſtruction.*

Thus therefore the whole, or at leaſt the chief buſineſs of Analyticks, is converſant in finding a convenient or fit Equation: For Reduction is very eaſie, and conſiſting only in eaſie Operations and mere Axioms, as *e. g.*

If to equal quantities you add or ſubſtract equal ones, the Aggregates or Remainders will be equal;

If equal quantities are multiplyed or divided by the ſame, the Products or Quotients thence ariſing will be equal, *&c.*

IV. *EFFECTION*, or *CONSTRUCTION.*

1. *In ſimple Equations.*

SUppoſe $z = b$, the quantity b is ſought.

2. If z be $= \frac{ab}{c}$ …. as c to b ſo a to z,
or $x = \frac{aa}{b}$ … as b to a ſo a to x,
make
or $y = \frac{fb + fg}{b - 1}$ as $b - 1$ to $b + g$ ſo f to y,
or $z = \frac{fb - fg}{b + 1}$ as $b + 1$ to $b - g$ ſo f to z, &c. every where according to *n.* 2. Schol. 2. Prop. 34. Lib. 1. *Math. Enuc.*

3. If z be $= \frac{kl + mn}{r + s}$, the Reſolution of it into Proportionals will be more difficult, becauſe neither of the Letters are found twice in the Numerator. That therefore you may have *e. g.* k twice, you muſt make as k to n ſo m to a fourth Proportional which call p; then will, by vertue of *Prop.* 18. *Lib.* 1. $kp = mn$, and the propoſed Equation be changed into this $z = \frac{kl + kp}{r + s}$, to be now conſtructed from the 2*d.* Caſe.

Or if $y = \frac{kl + mn}{r - s}$, find a mean Proportional between k and l, which call p; and between m and n, which call q, according to *n.* 3. of the afore-cited Schol.; and the propoſed Equation, by virtue of *Prop.* 17. will be in this form: $y = \frac{pp + qq}{r - s}$. Make therefore in the right-angled △ (*Fig.* 1.) AB $= p$ and BC $= q$; and the □ AC by vertue of the Pythagorick Theorem, $= pp + qq$: Which ſince it muſt be divided by $r - s$, make further, by *Prop.* 18 as $r - s$ to the $\sqrt{pp + qq}$, ſo is $\sqrt{pp + qq}$ to y, according to the afore-cited.

4. In like manner if x be $= \frac{bg - mn}{c + d}$, make 1*ſt.* as b to m ſo n to a fourth which call k; and ſo putting bk for mn, the Equation will be reduced to the ſecond Caſe under this form: $x = \frac{bg - bk}{c + d}$

Or thus: Find a mean Proportional between b and g, which call p, and between m and n, which call q; and the propoſed Equation will be in this Form: $x = \frac{pp - qq}{c + d}$ Make therefore (in *Fig.* 2.) $AB = p$, and having on this deſcribed a Semi-circle, apply $BC = q$; then will, by vertue of *Schol.* 5. *Prop.* 34. $\square AC = pp - qq$: Which ſince it muſt be divided by $c + d$, make farther,

as $c + d$ to $\sqrt{pp - qq}$, ſo $\sqrt{pp - qq}$ to x; all from the ſame Foundations, whence you have the Conſtruction of the third Caſe.

5. If z be $= \frac{aabc}{ffg}$; make firſt as f to a, ſo a to a third Proportional m, and you'l have (putting fm for aa) $z = \frac{fmbc}{ffg}$, i. e. $\frac{mbc}{fg}$. Make ſecondly as f to m, ſo b to a fourth n, and by putting fn for mb you'l have $z = \frac{fnc}{fg}$, i. e. $\frac{nc}{g}$, wherefore thirdly you'l have as g to n ſo c to z.

6. If y be $= \frac{hll}{mm}$, make firſt as m to n, ſo l to a fourth, which call n, and by putting now mn for hl, you'l have $\frac{mnl}{mm}$ i. e. $\frac{nl}{m} = y$. Therefore you'l now have ſecondly, as m to n, ſo l to y by Caſe 2: So that the Conſtruction of the fifth and ſixth Caſes is nothing but reiterations of the Rule of three, according to what we have often inculcated, *N.* 2. and 3. *Schol.* 2. *Prop.* 34.

2. *In ſimple Quadratick Equations.*

1.

$$\text{IF}\ \left.\begin{array}{l} xx = ab \\ \text{or } y^2 = 1c \\ \text{or } z^2 = \frac{3}{4}dd \end{array}\right\}\ \text{you'l have}\ \left\{\begin{array}{l} x = \sqrt{ab} \\ y = \sqrt{1c} \\ z = \sqrt{\frac{3}{4}dd} \end{array}\right.\ \text{that is to a mean proportional between}\ \left\{\begin{array}{l} a \text{ and } b \\ 1 \text{ and } c \\ \frac{3}{4}d \text{ and } d \end{array}\right.$$

and ſo the Conſtruction will be had from *n.* 3. *Schol.* 2. *Prop.* 34. (ſee Fig. 3.)

2. $$\text{If}\ \left.\begin{array}{l} y^2 = fg + kl \\ \text{or } x^2 = fg - kl \end{array}\right\}\ \text{you'l have}\ \left\{\begin{array}{l} y = \sqrt{fg + kl} \\ x = \sqrt{fg - kl} \end{array}\right.\ \text{make therefore}$$

on the one ſide the Right-angled Triangle ABC (*Fig.* 4.) whoſe ſide

AB is = to a mean Proportional between f and g,

BC is = to a mean Proportional between k and l;

On the other a Right-angled △(*Fig.* 5) whoſe ſide AB is = to a mean Proportional between f and g, and the ſide BC = to a mean Proportional between k and l;

$$\left.\begin{array}{l} \text{and on the one hand} \\ \text{the Hypothenuſa,} \\ \text{on the other the ſide} \end{array}\right\}\ \text{AC will be the value of}\ \left\{\begin{array}{l} y \\ \\ x \end{array}\right.\ \text{ſought.}$$

And all by vertue of the Pythag. Theor. and according to *Schol.* 5. *Prop.* 34. or the Confectarys of *Prop.* 44. See Fig 4. and 5.

3. If $z^2 = \frac{fhhkk}{fcc}$ i. e. $\frac{hhkk}{cc}$; extracting the Roots on both ſides, z will $= \frac{hk}{c}$, and ſo be the ſecond caſe of ſimple Equations.

4. If y^2 be $= \frac{fghk}{lm}$, make firſt as l to f, ſo g to a fourth which call n, and by putting ln for fg, you'l have $y^2 = \frac{lnhk}{lm}$ i. e. $\frac{nhk}{m}$. Make Secondly, as m to n, ſo h to a fourth, which call p; and by putting mp for nh, you'l have $y^2 =$

$\frac{mpk}{m}$ i. e. pk, and ſo the firſt caſe of the preſent Equations.

5. If $x^2 = \frac{qbcc + bccd + qbcd + bcdd}{fg + lm}$ in the firſt place the Rectangles fg and lm being turned into Squares, and collected into one Sum, make them $= nn$. Then (ſince cc and cd are multiplyed by $qb + bd$) in like manner qb and bd added make pp; and you'l have $x^2 = \frac{ppcc + ppcd.}{nn}$ Thirdly (ſince pp is already multiplyed by $cc + cd$) having added $cc + cd$ into one Sum that they may *e. g.* make rr; x^2 will $= \frac{pprr}{nn}$, and ſo be the third caſe of the preſent Equations.

3. *In affected quadratick Equations.*

1. If zz be $= az + bb$, then will $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$; which may be thus in ſhort demonſtrated *a priori*: Since $z^2 - az = bb$ *per Hypoth.* and that firſt quantity if it be added to $\frac{1}{4}aa$, it becomes an exact Square, the root whereof is $z - \frac{1}{2}a$; therefore $z^2 - az + \frac{1}{4}aa = bb + \frac{1}{4}aa$, and conſequently $z - \frac{1}{2}a = \sqrt{bb + \frac{1}{4}aa}$; and laſtly $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$ or $\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb}$; which laſt Root is a falſe one and leſs than nothing, but yet gives you the propoſed Equation back again as well as the former; as will be evident to any one who trys, *viz* having transferr'd $\frac{1}{2}a$ on the other ſide, and ſo the two equal quantities $z - \frac{1}{2}a$ and $-\sqrt{\frac{1}{4}aa - bb}$ being ſquared. For here will come out $\frac{1}{4}aa + bb$ as well as if the radical Sign were affected with the Sign $+$ becauſe $-$ by $-$ gives $+$. Therefore $zz - az + \frac{1}{4}aa = \frac{1}{4}aa + bb$, and taking away on both ſides $\frac{1}{4}aa$, $zz - az = bb$ i.e. $zz = az + bb$.

The value therefore of this Root will be had geometrically, by making (in *Fig.* 6.) $CD = \frac{1}{2}a$ and $DE = b$, that the Hypothenuſa CE may be $\sqrt{\frac{1}{4}aa + bb}$; and moreover, drawing out on both ſides CD, and at the interval CE deſcribing a Semi-circle mi-circle AEB: This being done, AD will be the value ſought of the true Root z, and DB of the falſe one.

2. If y^2 be $= -ay \dagger bb$, then will $y = -\frac{1}{2}a \dagger \sqrt{\frac{1}{4}aa \dagger bb}$; which again may thus appear: Since $y^2 \dagger ay$ is $= bb$ *per Hypoth.* adding to both ſides $\frac{1}{4}aa$, the firſt quantity will be an exact Square, and $y^2 \dagger ay \dagger \frac{1}{4}aa = \frac{1}{4}aa \dagger bb$. Therefore the Roots will be alſo equal, *viz.* $y \dagger \frac{1}{2}a = \sqrt{\frac{1}{4}aa \dagger bb}$, and conſequently $y = \sqrt{\frac{1}{4}aa \dagger bb} - \frac{1}{2}a$ or $-\sqrt{\frac{1}{4}aa \dagger bb} - \frac{1}{2}a$; which is a falſe Root.

The value of theſe Roots may be had geometrically, *viz.* of the true Root DB in *Fig.* 6. or BE in *Fig.* 7. and of the falſe one in the firſt AD, in the ſecond AE.

3. If xx is $= ax - bb$, you'l have

$$x = \tfrac{1}{2}a \dagger \sqrt{\tfrac{1}{4}aa - bb}$$
$$\text{or } \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bb}:$$

Which may be demonſtrated after the ſame way *a priori*, as the former Caſes, *viz.* Since $x^2 - ax$ is $= -bb$, adding on both ſides $\frac{1}{4}aa$, the former quantity will be an exact Square, *viz.* $x^2 - ax \dagger \frac{1}{4}aa = \frac{1}{4}aa - bb$. Therefore the Root of the one $x - \frac{1}{2}a =$ to the Root of the other, *viz.* $\sqrt{\frac{1}{4}aa - bb}$, and adding on both ſides $\frac{1}{2}a$, $x = \frac{1}{2}a \dagger \sqrt{\frac{1}{4}aa - bb}$ which is one of the true Roots. Or $\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$ which in this caſe is alſo a true one. But the value of each may be obtain'd by making (*Fig.* 8.) CB $= \frac{1}{2}a$ and by erecting BD perpendicularly $= b$, and making the Semi-circle BEA, and drawing DE parallel to CB, and letting fall the Perpendicular EF: For thus CF will be $\sqrt{\frac{1}{4}aa - bb}$ and conſequently AF, $\frac{1}{2}a \dagger \sqrt{\frac{1}{4}aa - bb}$, and FB $\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$.

Or, with *Cartes*, making (*Fig.* 9) CB $= \frac{1}{2}a$ and BD $= b$, drawing DF parallel to CB, that FD may be one root and ED the other; as is manifeſt from the precedent Conſtruction, and its Rule. See alſo another Deduction from *Cartes*'s Conſtructions, *Schol.* 1. *Prop.* 47. *Lib.* 1. *Math. Enucl.*

NB. 1. The ingenious *Schooten* has before ſhew'd this Method of demonſtrating, and alſo of finding out theſe Rules in his Comment on the Geometry of *Des Cartes*, *p. m.* 163. and moreover deduces another ingenious Method for all the three Caſes of theſe Equations, by taking away the ſecond term in the Equation, *p.* 290, and the following, where we may make only this Remark concerning the third Caſe; that perhaps the Rule might be better deduc'd, if we make $x = \frac{1}{2}a - z$ rather than $x = z - \frac{1}{2}a$.

NB. 2. If any one has a mind to ſee the new Conſtructions of affected quadratick Equations of the Abbot *Catelan*, he may find them in *Acta Erud. Lipſ. Ann.* 1682. p. 86. and in the 27*th* Journal *des Sçavans.* 1 *Dec.* 1681.

IV. *For Cubick and Biquadratick Equations both ſimple and affected, and alſo for all before mentioned, and conſequently univerſally for all not exceeding the fourth Dimenſion.*

THE value of the unknown Quantity or Root may for any Caſe be determined by one general Rule, found out by Mr. *Thomas Baker* an Engliſh-man, occaſioned by what *Des Cartes* had taught concerning this matter, *Lib.* 3. *Geom. p.* 85, and the following; but now very much perfected by this Rule, and made more ſimple. Now that this Rule may be the better comprehended by Learners, we will premiſe theſe following things.

1. That all Equations occurring under thoſe Forms which we have before ſhewn in the Article of Reduction, or the like, muſt always for this purpoſe be ſo changed as to have all the terms or parts of the Equation both known and unknown, affected and not affected, brought over to one ſide promiſcuouſly, and ſo on the other there will ſtand *o* or nought, as *e g.* let $z - b$ be $= 0$, or $y - \frac{ab}{c} = 0$, or $z^3\ abc = 0$, or

$$y^3 - \frac{a^3bb}{ca} = 0, \text{ or } x^2 + ax - b^2 = 0; \text{ or}$$

$$y^2 - 2by - \frac{2abc}{d} = 0; \text{ or}$$

$$z^3 - az^2 - b^2z + c^3 = 0; \text{ or}$$

$$y^4 - ay^3 - b^2y^2 + c^3y - d^4 = 0 \text{ \&c.}$$

which alſo was uſual to *Des Cartes* in *Lib.* 3.

2. In all Equations the known quantity or Co-efficient of the ſecond Term we will generally denote by the Letter p, that of the third Term by the Letter q, of the fourth by r, and the fifth (or abſolute Number) by S; according to *Cartes*, but with ſome little alteration: So that hence the Equations we have before been treating of, and all others like them (every where denoting the unknown quantity by x) may all be reduced to theſe forms:

$x - p = 0$

$x^2 + px - q = 0$

$x^3 - px^2 - qx + r = 0$

$x^4 - px^3 + qx^2 - rx + S = 0$, &c. &c.

3. Theſe and the like Equations may either occur whole, or with all their Terms, as here, or depriv'd of one or more of them, as the following Examples will ſhew, where we will always put an Aſterisk in the place of the deficient Term.

$x^2 \ast - q = 0$

$x^3 \ast\ast + r = 0$

$x^4 \ast\ast\ast - S = 0$

$x^3 \ast + qx \pm r = 0$

$x^4 \ast \pm qx^2 + rx \pm S = 0$

$x^4 + px^3 \pm qx^2 \ast + S = 0$

$x^4 \ast \ast + rx \pm S = 0$

$x^4 + px^3 \ast \ast \pm S = 0$

4. The unknown quantity in any Equation has † as many diverſe Roots or Values, as the Equation has Dimenſions; which *Des Cartes* ſhews, *Lib.* 3. *Geom.* p. 69. at the ſame time evidently demonſtrating this, *viz.* that ſome of thoſe Roots may be falſe ones, *i. e.* leſs than nothing: Which from him we here ſuppoſe.

† N. *This is an error, as has been prov'd by Dr.* Wallis.

When therefore *Des Cartes* in his Conſtruction of Cubick and biquadratick Equations, *p.* 85, and the following, requires as a neceſſary Condition, the ejection of the ſecond Term in the given Equation, unleſs it were already wanting, and ſo was obliged to ſhew a way to eject it, with ſeveral other Preparations; and afterwards, when by help of his Rule delivered *p.* 91. he had found a way of finding two mean Proportionals, and dividing any given Angle into three equal parts, then he uſes it for ſolving other ſolid Problems, or finding two mean Proportionals, or triſecting an Angle. But the general Rule of *Baker* has no need of theſe methods or helps, neither of the Ejection of the ſecond Term, nor any other Preparation, but immediately ſhews us a way, by the help of a Circle and Parabola to find all the Roots of any given Equation, both true and falſe, whether the Equation want any term or not, and howſoever affected, after the way we will now, and perhaps a little more diſtinctly, ſhew.

1. It ſuppoſes with *Cartes* a Parabola NAM to be already deſcribed, (See *Fig.* 10. and 11.) whoſe *Latus Rectum* ſhall be L or 1, and its Axe ay; which *Des Cartes* only making uſe of, and never thinking of the other Diameters, was forced to take away the ſecond Term of the Equation, *&c.* *Baker* therefore (ſtrangely perfecting the *Carteſian* Geometry by this one thought) if the quantity p or ſecond Term be in the Equation applys to the Ax ay (or draws an ordinate to it) $BA = \frac{p}{2}$ i.e. he erects at top of the Ax a on the right hand the perpendicular $aE = \frac{p}{4}$ and from E draws EAy parallel to the Ax ay; whereby he obtains the Diameter Ay ſought.

2. Having made this Preparation, the whole buſineſs depends on this, to find the Center of the Circle to be deſcribed through the Parabola, which (by vertue of ſome arbitrary ſuppoſitions in the beginning) he always ſeeks on the left ſide of the Ax or Diameter, by help of two Lines $\genfrac{}{}{0pt}{}{AD}{aD}$ or b, and DH or d; *viz.* by placing the former upon the Ax from a to D, if p be wanting in the Equation, or upon the Diameter Ay from A to D if p be there; and letting fall from the point D the latter to aD or AD perpendicularly towards the left hand.

3. He ſhews how to find the quantity of either of theſe Lines (which is here very requiſite) in any given Equation, by a certain general Rule (which he calls the Central Rule, becauſe it alone helps to find the Center H) comprehended in theſe terms:

A SYNOPSIS *of Mr.* Baker's CLAVIS, *to be annexed to* Page 13, *of the* Introduction *to the* Specious Analyſis.

Of Æquations.	*Of Æquations.*	*Of Central Rules.*
Claſſ. I.	I. $x^4 *** - S = 0$	$\frac{1}{2} = b = AD,\ 0 = d = DH.$
II. 1. $x^2 * - q = 0$ 2. $x^2 * + q = 0$ *Impoſſible.*	1, 3 $x^4 * - qx^2 * \mp S = 0$ 2, 4 $x^4 * + qx^2 * \mp S = 0$	$\left.\begin{matrix}\frac{1}{2} + \frac{q}{2}\\ \frac{1}{2} \backsim \frac{q}{2}\end{matrix}\right\} = b = AD,\ 0 = d = DH$
III. 1. $x^3 ** - r = 0$ 2. $x^3 ** + r = 0$	1, 3 $x^4 ** - rx \mp S = 0$ 2, 4 $x^4 ** + rx \mp S = 0$	$\frac{1}{2} = b = AD$ $\frac{r}{2} = d = DH$
1. $x^3 * - qx - r = 0$ 2. $x^3 * - qx + r = 0$	1, 3 $x^4 * - qx^2 - rx \mp S = 0$ 2, 4 $x^4 * - qx^2 + rx \mp S = 0$	$\frac{1}{2} + \frac{q}{2} = b = AD$ $\frac{r}{2} = d = DH$
IV. 3. $x^3 * + qx - r = 0$ 4. $x^3 * + qx + r = 0$	5, 7 $x^4 * + qx^2 - rx \mp S = 0$ 6, 8 $x^4 * + qx^2 + rx \mp S = 0$	$\frac{1}{2} \backsim \frac{q}{2} = b = AD$ $\frac{r}{2} = d = DH$
$\frac{p}{2} = BA$	$\frac{p}{2} = BA$	$\frac{p}{4} = AE$ *in the* 5, 6, 7, 8 *Claſs.*
V. 1. $x - p = 0$ 2. $x + p = 0$	1, 3 $x^4 - px^3 ** \mp S = 0$ 2, 4 $x^4 + px^3 ** \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} = d = DH$
1. $x^3 - px^2 * + r = 0$ 2. $x^3 + px^2 * - r = 0$	1, 3 $x^4 - px^3 + rx \mp S = 0$ 2, 4 $x^4 + px^3 * - rx \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{r}{2} = d = DH$
VI. 3. $x^3 - px^2 * - r = 0$ 4. $x^3 + px^2 * + r = 0$	5, 7 $x^4 - px^3 * - rx \mp S = 0$ 6, 8 $x^4 + px^3 * + rx \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{r}{2} = d = DH$
1. $x^2 - px - q = 0$ 2. $x^2 + px - q = 0$	1, 3 $x^4 - px^3 - qx^2 * \mp S = 0$ 2, 4 $x^4 + px^3 - qx^2 * \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} + \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} = d = DH$
VII. 3. $x^2 - px + q = 0$ 4. $x^2 + px + q = 0$	5, 7 $x^4 - px^3 + qx^2 * \mp S = 0$ 6, 8 $x^4 + px^3 + qx^2 * \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} \backsim \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{pq}{4} = d = DH$
1. $x^3 - px^2 - qx - r = 0$ 2. $x^3 + px^2 - qx + r = 0$	1, 3 $x^4 - px^3 - qx^2 - rx \mp S = 0$ 2, 4 $x^4 + px^3 - qx^2 + rx \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} + \frac{q}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} + \frac{r}{2} = d = DH$
3. $x^3 - px^2 - qx + r = 0$ 4. $x^3 + px^2 - qx - r = 0$	5, 7 $x^4 - px^3 - qx^2 + rx \mp S = 0$ 6, 8 $x^4 + px^3 - qx^2 - rx \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} + \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} = d = DH$
VIII. 5. $x^3 - px^2 + qx + r = 0$ 6. $x^3 + px^2 + qx - r = 0$	9, 11 $x^4 - px^3 + qx^2 + rx \mp S = 0$ 10, 12 $x^4 + px^3 + qx^2 - rx \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} \backsim \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{pq}{4} \backsim \frac{r}{2} = d = DH$
7. $x^3 - px^2 + qx - r = 0$ 8. $x^3 + px^2 + qx + r = 0$	13, 15 $x^4 - px^3 + qx^2 - rx \mp S = 0$ 14, 16 $x^4 + px^3 + qx^2 + rx \mp S = 0$	$\frac{1}{2} + \frac{p^2}{8} \backsim \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{pq}{4} + \frac{r}{2} = d = DH.$

N B. *The Sign ∽ denotes a dubious Caſe,* viz. *That either the Antecedent muſt be Subſtracted from the Conſequent, or the Conſequent from the Antecedent, according as the matter will bear.*

$$\text{Part}\begin{cases} 1.\ \frac{L}{2} \dagger \frac{p^2}{8L} \overset{\dagger}{\backsim} \frac{q}{2L} = b = \begin{matrix}AD\\ aD\end{matrix} \text{ or} \\ 2.\ \frac{p}{4} \dagger \frac{p^3}{16L^2} \overset{\dagger}{\backsim} \frac{pq}{4L^2} \overset{\dagger}{\backsim} \frac{r}{2L^2} = d = DH. \end{cases}$$

4. This Rule as it ſtands here whole, only anſwers to thoſe Equations wherein are all the Terms p, q, and r; and in the mean time may alſo be eaſily accommodated to all other Caſes, only obſerving theſe things. 1. Whatever Term, or Quantities p, q, r, be wanting in the propoſed Equation, that muſt alſo be reſpectively omitted, or put out of the general Central Rule, that the remaining quantities may determine the ſpecial or particular Central Rule. 2. As for what belongs to the Signs, *viz.* whether $\dagger$ or $\backsim$ (which latter Sign denotes a dubious Caſe, either that the firſt muſt be ſubſtracted from the latter, or contrary-wiſe, as the matter will bear) muſt be put in the Central Rule, he notes (α) that in the Rule you'l always have $\frac{\dagger\, r,}{2L^2}$ unleſs when in the propoſed Equation p and r are affected with diverſe Signs: (β) By what Sign ſoever in the propoſed Equation it happens that the quantity q is marked with, it muſt be noted with the contrary one (altho' involv'd with other quantities) in the Rule; as may be ſeen in the application of the Rule to all ſpecial Caſes done by the Author himſelf for the ſake of Beginners, and is exhibited in the Synopſis hereunto adjoyning, which yet we have thought fit to give at the end of this Treatiſe, much more contract as to the Central Rules, in a ſhort Compendium by way of Appendix.

5. By theſe Rules therefore, the quantities of the Lines aD or AD and DH will be ſo determined, that the parts in the Rule marked with the Sign $\dagger$ (taken either aggregately or ſingly) will be put downwards from a to A towards y, and on the left hand of D; but the negative Parts, or thoſe affected with the Sign——, will be cut off, on the one part above, on the other on the right hand: Which being done the Center H will be found.

6. From the Center H thro' the Vertex of the Ax a (if the point D is found in the Ax) or in the other Caſe thro' the Vertex of the Diameter A, you muſt draw a Circle which by cutting or touching the Parabola will deermine the Roots

ſought, if the Equation be not a Biquadratick *i. e.* has not the quantity S; otherwiſe another Point L or Z muſt be found, (*vid.* Fig. 12. and 13.) and a Circle deſcribed on the Radius HL or HZ, according to *Des Cartes* p. 86, and following, of his Geometry.

7. *Viz.* If you have——S, you muſt take on the Line H*a* or HA produced, on the one ſide AI = L or 1, and on the other AK = $\frac{S,}{L^3}$ and deſcribing a Semi-circle on IK, draw AL perpendicular to AH, to obtain the point L. (ſee Fig. 12.) But if you have † S, then in another Semicircle deſcribed on AH, apply the Line AZ = to AL found, thereby to obtain Point Z, (ſee Fig. 13.)

8. A Circle therefore deſcribed from H through *a* or A, if S be wanting, but thro' L if there be——S, and thro' Z if † S, may touch or cut the Parabola either in 1, 2, 3 or 4 Points; from which if you let fall Perpendiculars to the Ax or Diameter, you will obtain all the Roots of your Equation both true and falſe.

9. And, 1. If in the Equation *p* be wanting and—*r* be there, the true Roots will be on the left ſide of the Ax, as NO, and the falſe ones as MO on the right ſide. 2. But if there be in the Equation *p* and—*p*, the true Roots will fall on the left ſide of the Diameter, and the falſe ones on the right; but if † *p*, on the contrary the true will be on the right hand and the falſe on the left.

10. But if the Circle neither touches nor cuts the Parabola in any point, it is a ſign that the Equation is impoſſible, and has no Root either true or falſe, but only imaginary ones. All which, how they may be found out, and that they are undoubtedly true, are demonſtrated *a poſteriori*, in an eaſie and plain way by the Author, wherefore we ſhall not give the Demonſtrations of them here; but remit the Reader, after he has made a little progreſs in this Art, to the Author himſelf.

11. Wherefore now, (omitting alſo in this place the Doctrine of the Compoſition of the plain and ſolid Geometrical *Loci*, or Places, which would ſerve for a Complement of the Analytick Art) we will ſhew the Practice of theſe Rules already delivered, premoniſhing only this from Mr. *Baker*, if the *Latus Rectum* be made Unity, that L in the Central Rules

and all its Powers may be omitted, and ſo the Rules exhibited more compendiouſly, as we have already done in our Synopſis, and may be ſeen from the form of a general Central Rule hereunto annexed.

$$\text{Part}\begin{cases} 1.\ \frac{1}{2} \dagger \frac{p^2}{8} \underset{\backsim}{\dagger} \frac{q}{2} = b = \begin{matrix} a\mathrm{D} \\ \mathrm{AD} \end{matrix}\ \text{or} \\ 2.\ \frac{p}{4} \dagger \frac{p^2}{16} \underset{\backsim}{\dagger} \frac{pq}{4} \underset{\backsim}{\dagger} \frac{r}{2} = d = \mathrm{DH}. \end{cases}$$

To which Premonition of *Baker* we may alſo add this, if any given Line in the Problem it ſelf be taken for Unity, which may be often very commodiouſly done [as *a* in the former Problem p. 91. *Geom. Cartes*, and the Line NO in the latter, and *a* again in the Equation p. 83. the laſt line] and then the ſame Line alſo may be taken for the *Latus Rectum* of the Parabola to be deſcribed, if we have a mind to make uſe of this Compendium for abbreviating the Central Rules. For otherwiſe if we would conſtruct all Problems, as *Baker* rightly aſſerts we may, by only one Parabola, we ſhall fall often into very tedious Prolixities.

SOME EXAMPLES OF SPECIOUS ANALYSIS,

In each kind of Equations.

I. *In Simple Equations.*

PROBLEM I.

Having the ſum of any two ſides given for forming a Triangle ABC, to find each of the ſides, and form the Triangle.

Supppoſe *e. g.* three Lines given in *Fig.* 14. the firſt $= AB \dagger AC$ in the Triangle ſought, the ſecond $= AB \dagger BC$, the third $= BC \dagger AC$, to find each of the ſides *e. g.* to find AB, which being known, the reſt will be ſo alſo.

SOLUTION.

1. *Denomination.* Make $AC \dagger AB = a$; $AB \dagger BC = b$; $BC \dagger AC = c$; $AB = x$; then will $AC = a - x$, and $BC = b - x$, and ſo the Denomination be compleat.

2. *Equation.* Now if the values of the two laſt Lines BC and AC be added into one Sum, which we had before given; you'l have this Equation $a \dagger b - 2x = c$.

3. *Reduction.* By adding on both ſides $2x$, you'l have $a \dagger b = c \dagger 2x$; and ſubſtracting from both ſides c .. $a \dagger b - c = 2x$; and dividing both ſides by 2 ... $\frac{a \dagger b - c}{2} = x$.

4. The *Effection* or *Geometrical Conſtruction*, which the Equation thus reduced will help us to

Join $AE = a$ and $ED = b$ in one Line AD, and from this backwards cut off $DF = c$; and divide AF which remains into two equal parts in B, and you'l have AB the firſt ſide of the Triangle to be formed; and BE will give the other ſide AC, which ſubſtracted from ED, will leave GD = to the third ſide BC; of which you may now form the Triangle ABC.

5. *A general Rule for Arithmetical Caſes.* Add the two former Sums, and from the Aggregate ſubſtract the third Sum; half the Remainder will give the ſide AB common to the two former Sums. For an Example take this Queſtion: There are three Towns of ancient *Hetruria*, viz. *Forum Caſſii* (which the Letter A denotes in △ ABC) *Sudertum* (B) and *Volſinii* (C) which are at this diſtance one from another; if you go from *Volſinii* to *Forum Caſſii* and thence to *Sudertum*, you mſt go 330 Furlongs; from *Forum Caſſii* to *Sudertum* and thence to *Volſinii* there are 306 Furlongs; laſtly, from *Sudertum* to *Volſinii* and thence to *Forum Caſſii* 272 Furlongs. How far is each Town diſtant from each other.

PROBLEM II.

In a right-angled Triangle ABC, having given the Baſe AB, and the difference of the Perpendicular AC and the Hypothenuſa BC to find the Perpendicular and Hypothenuſa, and form the Triangle.

Make *e. g.* the Baſe AB (*Fig.* 15.) and the difference of the Perpendicular and Hypothenuſa BD, to find the Perpendicular AC; which being known, the Hypothenuſa AC will be known alſo, if the given difference be added to the found Perpendicular.

SOLUTION.

1. *Denomination.* Make $AB = a$, $BD = b$, $AC = x$; then will $BC = x \dagger b$.

2. *Equation* by the Pythagorick Theorem,

$xx \dagger aa = xx \dagger 2bx \dagger bb$, viz. the two Squares of the Sides to the Square of the Hypothenuſa.

3. *Reduction.* Subſtracting from both ſides xx, you'l have $aa = 2bx + bb$; and moreover by ſubſtracting alſo bb, $aa - bb = 2bx$; and dividing by $2b$

$$\frac{aa-bb}{2b} = x.$$

4. *Effection* or *Geometrical Conſtruction.* Having deſcribed upon the given Baſe AB a Semi-circle, apply therein the given difference BD, and draw AD, whoſe Square is $= aa - bb$. Since this muſt be divided by $2b$, make, as AE $= 2b$ to AD $= \sqrt{aa - bb}$, ſo AD $= \sqrt{aa - bb}$ to AC the Perpendicular ſought. To which if you add CF $=$ BD, you will have AF $=$ to the Hypothenuſa ſought BC; which will come of courſe together with the whole Triangle ſought, if the found Perpendicular AC be erected at right Angles on the given Baſe AB.

5. *The Rule for Arithmetical Caſes.* From the ſquare of the given Baſe ſubſtract the ſquare of the given difference, and divide the Remainder by the double difference; and you'l have the Perpendicular ſought. *E. g.* ſuppoſe the Baſe $=$ 20 foot, and the difference between the Perpendicular and Hypothenuſa 10.

PROBLEM III.

IN the right angled Triangle ABC, having given the ſide AC and the ſum of the other ſide AB and the Hypothenuſa BC, to find the other ſide and the Hypothenuſa ſeparately, and form the Triangle. Suppoſe the given ſide (that is to be) AC (*Fig.* 16.) and the ſum of the other ſides AD, to find the ſide AB, which being known the Hypothenuſa BC willl be known alſo.

SOLUTION.

1. *Denomination.* Make AC $= a$, AD $= b$, AB $= x$, then will BC $= b - x$.

2. *Equation.* $xx + aa = bb - 2bx + xx$ and ſubſtracting xx.

3. *Reduction.* $aa = bb - 2bx$; and adding $2bx$,
$aa \dagger 2bx = bb$; and ſubſtracting aa,
$2bx = bb - aa$; and dividing by $2b$,
$$x = \frac{bb - aa}{2b}.$$

4. The *Effection* or *Conſtruction* is like the former, and ſo will be manifeſt only by inſpecting that Scheme.

5. *The Arithmetical Rule.* From the ſquare of the given ſum ſubſtract the ſquare of the given ſide, and divide the Remainder by double the given ſum; and youl l have the other ſide, and ſubſtracting that from the given ſum, you have the Hypothenuſa alſo. *E.g.* let one ſide be 15, and the ſum of the other two 45.

PROBLEM IV.

Having given the Perpendiculars and ſum of the Baſes of two right-angled Triangles having equal Hypothenuſes, to find the Baſes ſeparately, and form the Triangles. Suppoſe *e.g.* to form the Triangle ABC (ſee *Fig.* 17.) you have given the Perpendicular AB, and for the other △ ADC, the Perpendicular CD, and the given ſum of the Baſes BE, to find the Baſes ſingly, *viz.* the leſs for the greateſt Perpendicular, and the greater AD for the leſs Perpendicular.

SOLUTION.

1. *Denomination.* Make AB $= a$, CD $= b$, the ſum BE $= c$; make the leſſer Baſe BC $= x$; the greater AD will $= c - x$.

2. *Equation.* Since the Hypothenuſes of the two Triangles are ſuppoſed equal, the two $\square\square$ AB † BC, *i. e.* $xx \dagger aa$ will be $=$ to the two $\square\square$ AD † CD, *i. e.* $bb \dagger cc - 2cx \dagger xx$.

3. *Reduction.* By taking away therefore xx and adding $2cx$, $aa \dagger 2cx$ will $= bb \dagger cc$; and further taking away from both ſides aa, $2cx = bb \dagger cc - aa$; and dividing both ſides by $2c$, $x = \frac{bb \dagger cc - aa.}{2c}$

4. *Geometrical Conſtruction.* Join the Lines b and c *i. e.* CD and BE at right Angles, (*n.* 2) and the ſquare of the Hypothenuſa DE will $= bb + cc$. Upon this Hypothenuſa having deſcribed a Semi-circle, apply therein the Line AD, and the ſquare of AE will $= bb + cc - aa$. Which, ſince it muſt be further divided by $2c$, make (*n.* 3.) as BF $= 2c$ to BG $= \sqrt{bb + cc - aa}$, ſo BG to BC, the leſſer Baſe ſought.

5. *The Arithmetical Rule.* From the ſum of the ſquares of the leſſer Perpendicular and the ſum of the Baſes ſubſtract the ſquare of the greater Perpendicular, and the Remainder divided by the double ſum of the Baſes, will give the leſſer Baſe. *E. g.* make AB 76, CD 57, and BE 114.

PROBLEM V.

Having given the Perpendiculars of two right-angled Triangles ſtanding on the ſame given Baſe, to find the Segments of the Hypothenuſes. E. g ſuppoſe the common given Baſe be AB (*Fig.* 18.) and the Perpendicular of one Triangle AD of the other BC; to find the ſegments of the Hypothenuſes, cutting one another geometrically.

SOLUTION.

If a Geometrical Solution be required there is no need of any Analyſis; for having erected perpendicularly on the common given Baſe AB the given Perpendiculars AD and BC, the Hypothenuſes AC, BD being drawn immediately, exhibit their Segments EA, EB, EC, ED. But if it be to be done arithmetically by a general Rule, then an Analyſis will be neceſſary.

1. *Denomination.* Make the common Baſis AB $= a$, BC $= b$, AD $= c$; and, having found AF, all the reſt may be had (for as AB to BC ſo AF to FE; which being given, you have alſo GD and HC, and conſequently alſo DE, CE, *&c.*) make AF $= x$, then will BF or HE $= a - x$.

2. The *Equation* from FE found twice.

1. As AB to BC ſo AF to FE.

$$a \text{ —— } b \text{ —— } x \text{ —— } \frac{bx}{a}$$

Pag. 5.

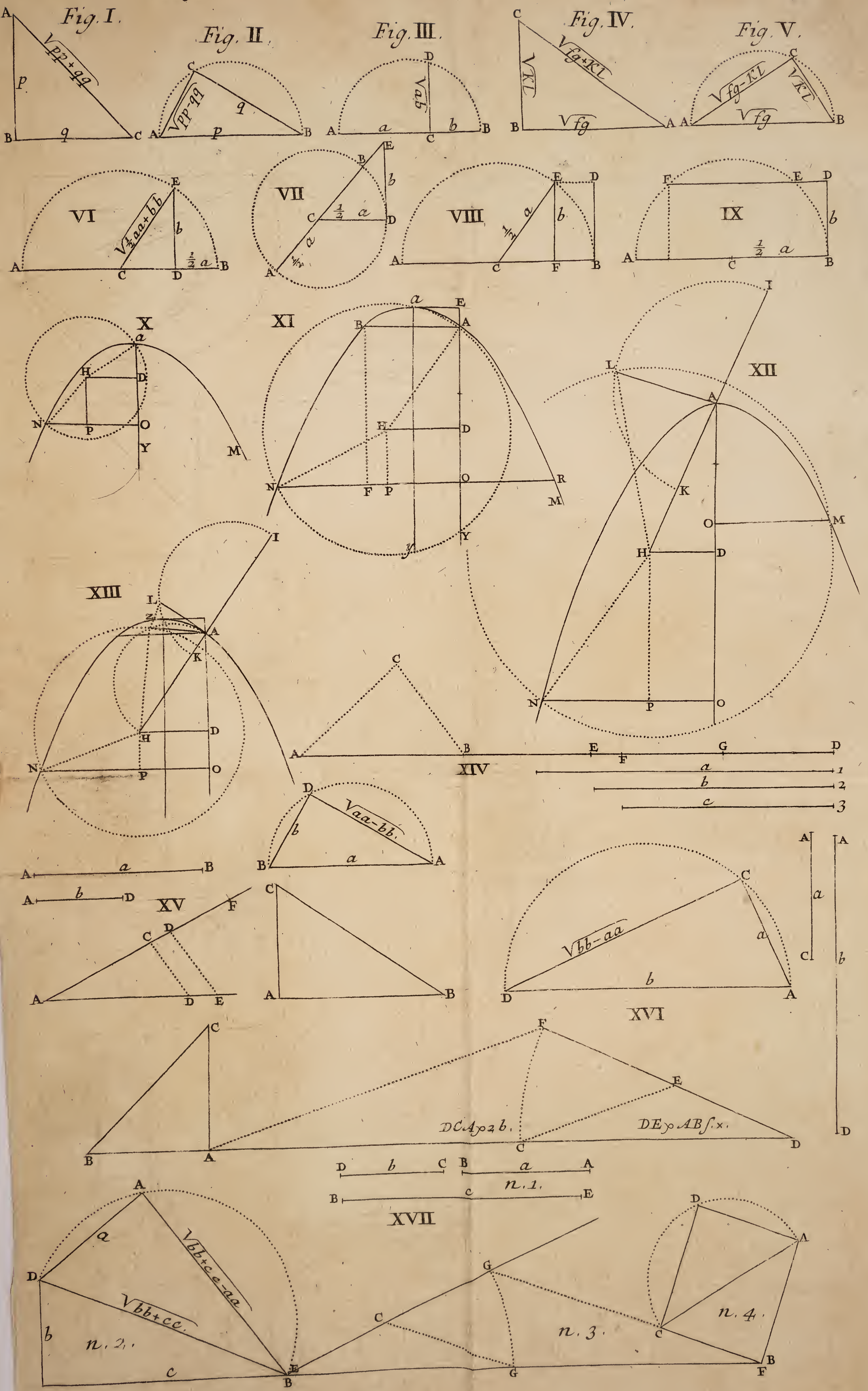
Fig. I.
A
√pp+qq
p
B
q
C
Fig. II.
C
√pp-qq
q
A
p
B
Fig. III.
D
√ab
A
a
C
b
B
Fig. IV.
C
√KL
√fg+KL
B
√fg
A
Fig. V.
C
√fg-KL
√KL
A
√fg
B
VI
E
√¼aa+bb
b
A
C
D
½ a
B
VII
B
E
b
C
½ a
D
½ a
A
VIII
E
D
½ a
b
A
C
F
B
IX
F
E
D
b
½ a
A
C
B
X
a
H
D
N
P
O
Y
M
XI
a
E
B
A
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y
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A
K
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M
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N
P
O
XIII
I
L
Z
A
K
H
D
N
P
O
XIV
C
A
B
E
F
G
D
a 1
b 2
c 3
D
b
√aa-bb.
B
a
A
A a B
A b D
XV
F
C
D
A
D
E
C
A
B
XVI
C
√bb-aa
a
D
b
A
A a C
A b D
C
B
A
F
E
DCA ∞ 2b.
DE ∞ AB ſ x.
C
D
D b C
B a A
n.1.
B c E
XVII
A
a
√bb+cc-aa
D
√bb+cc.
b
n.2.
c
E
B
C
G
G
n.3.
D
A
n.4.
C
B
F

2. As BA to AD ſo BF to FE.

$$a \text{ —— } c \text{ —— } a-x \text{ —— } \frac{ac-cx}{a}.$$

Therefore $\frac{bx}{a} = \frac{ac-cx}{a}$.

3. *Reduction.* Multiplying both ſides by a you'l have $bx = ac - cx$; and adding on both ſides cx, $bx + cx = ac$; and dividing both ſides by $b+c$, $x = \frac{ac}{b+c}$.

4. *The Arithmetical Rule.* Multiply the commom baſe by the leaſt Perpendicular, and divide the Product by the ſum of the Perpendiculars; and you'l have the leſſer ſegment of the baſe, which being given you'l have all the reſt. *E. g.* ſuppoſe AB $= 10$, BC $= 9$, AD $= 6$.

PROBLEM VI.

T*O inſcribe a Rhombus in a given Oblong,* i. e. *having the ſides of the Oblong AB and BC given,* (*Fig.* 19.) *to find the Segment BF or DE, which being cut off, the remainder FC or AE will be the ſide of the Rhomb ſought.*

SOLUTION.

Make AB $= a$, BC $= b$, BF $= x$: FC or FA will be $= b - x$ (ſo far the *Denomination.*) Therefore the ſquare of FA, which is $bb - 2bx + xx$ will be $= aa + xx$, *viz.* to the two ſquares of AB and BF (ſo far the *Equation*;) and ſubſtracting rom both ſides xx, $bb - 2bx = aa$; and

$bb = aa + 2bx$, and

$bb - aa = 2bx$; and

$\frac{bb-aa}{2b} = x$. (ſo far the *Reduction.*)

The Geometrical Conſtruction. Having deſcribed a ſemiircle on BC (*n* 2.) apply CD or AB, and the □ BD will be $= bb - aa$. Which ſince it muſt be divided by $2b$, make s BE $= 2b$ to BD $= \sqrt{bb-aa}$, ſo BD to BF ſought, and o be cut off the ſide of the Oblong BC (*n.* 1.)

The Arithmeticai Rule. From the ſquare of the greater ſide ſubſtract the ſquare of the leſſer, and divide the remainder by double the greater ſide, and the quotient will give the Segment BF ſought. *E. g.* ſuppoſe $AB = 4$, and $BC = 8$.

PROBLEM VII.

TO *inſcribe the greateſt ſquare poſſible in a given Triangle, i. e.* having given the heighth of the Triangle CD (*Fig.* 20.) and the Baſe AB, to find a portion of the altitude CE, which being cut off there ſhall remain $ED = FG$.

SOLUTION.

Make the baſe $AB = a$; the altitude $CD = b$, $CE = x$; then will ED or $FG = b - x$.

By reaſon of the ſimilitude of the Triangles ABC and FGC you'l have as AB to CD ſo FG to CE.

$$a - b - b\cdot x - x$$

Therefore the Rectangles of the means and extremes will be equal, *i. e.* $ax = bb - bx$; and adding on both ſides bx, $ax \dagger bx = bb$, and dividing by $a \dagger b$, $x = \dfrac{bb}{a \dagger b}$

Conſtruction. Upon the ſide of the Triangle CB produced, make $CH = b$, and $HI = a$, ſo that the whole Line ſhall be $a \dagger b$. And having joined ID and parallel to it HE drawn from H, the part CE will be cut off, which is that ſought.

For as CI to CD ſo CH to CE,

$a \dagger b - b - b - x$ according to the ſecond caſe of ſimple Effections.

Arithmetical Rule. Square the given heighth of the Triangle, and divide the Product by the ſum of the baſe and altitude; and the quote is the part to be cut off CE. *E. g.* ſuppoſe $CD = 10$, and $AB = 15$.

PROBLEM VIII.

IN *an acute-angled Triangle having all the ſides given, to find the Perpendicular that ſhall fall from the Vertex on the Baſe,* i. e. having given AB, AC, BC (*Fig.* 21.) to find AD or BD (for having found the one you may eaſily find the other) *Coroll. Prop.* 13. *Lib.* 2. *Eucl.*

SOLUTION.

If there be only required a Geometrical Conſtruction of this Problem, there will be no need of any Analyſis; for having formed a Triangle ABC of the three given ſides, you need only let fall the perpendicular AD from the Vertex A, which would determine the Segment BD. To find the general Arithmetical Rule, which is the general Corollary of *Euclid*, or if any one for exerciſe ſake had rather determine the Perpendicular DA by the ſegment of the Baſe BD, than the latter by the former, the Analyſis will proceed thus:

Make AB $= a$, BC $= b$, AC $= c$, BD $= x$; then will CD be $= b - x$. Wherefore by the Pythagorick Theorem $\square$ AD $= aa - xx$, and by the ſame reaſon the ſame $\square$ AD $= cc - bb + 2bx - xx$. Therefore $ax - xx = cc - bb + 2bx - xx$; and by adding on both ſides xx, $aa = cc - bb + 2bx$; and by transferring $cc - bb$, $aa - cc + bb = 2bx$, and dividing by $2b$, $\frac{aa - cc + bb}{2b} = x$.

The Arithmetical Rule. Subſtract the ſquare of the leſſer ſide from the Sum of the $\square\,\square$ of the baſe and greater ſide, and the remainder divided by double the baſe will give its greater ſegment: If the $\square$ of the greateſt ſide be ſubſtracted from the ſum of the other ſquares, *&c.* you will have the leſs ſegment CD.

Geometrical Conſtruction Having deſcribed a ſemi circle upon AB (*n.* 2.) apply therein AC, and the $\square$ BC will $= aa - cc$; and continuing AC to 𝔅, till C𝔅 be $=$ CB (*n.* 1.) the $\square$ of 𝔅B will $= aa - cc + bb$; which ſince it is to be divided by $2b$ make as 𝔅E $= 2b$ to 𝔅B $= \Upsilon\, aa - cc + bb$, ſo 𝔅F $=$ 𝔅B to 𝔅D the ſegment ſought $=$ BD *n* 1.

PROBLEM IX.

IN *an obtuſe angled Triangle having the thr e ſides given to find the Perpendicular let fall from the Vertex to the Baſe being continued:* i. e. Having given AB, BC, AC (*Fig.* 22. *n.* 1.) to find AD or CD (for the one being found the other will readily be ſo alſo) *Coroll Prop.* 12. *Lib.* 2 *Eucl.*

SOLUTION.

What we premoniſhed about the former Problem, we underſtand to be premoniſh'd here alſo. For the reſt make here alſo $AB = a$, $BC = b$, $AC = c$, $CD = x$; then will $BD = b + x$: Wherefore by the Pythagorick Theorem the □ AD will $= cc - xx$, and by the ſame Theorem the ſame □ AD $= aa - bb - 2bx - xx$.

Therefore $cc - xx = aa - bb - 2bx - xx$; and adding to both ſides $cc = aa - bb - 2bx$; and tranſpoſing cc and $-2bx$, $2bx = aa - cc - bb$; and dividing by $2b$, $x = \frac{aa - cc - bb.}{2b}$

The Arithmetical Rule. Subſtract from the ſquare of the greater ſide the ſum of the ſquares of the baſe and leſſer ſide; and the Remainder divided by double the baſe will give its continuation to the Perpendicular.

Geometrical Conſtruction from the Equation reduc'd: Having deſcribed a ſemi-circle upon AB (*n.* 2.) apply therein AC, and the □ of CB drawn will $= aa - cc - bb$; which ſince it muſt be divided by $2b$ make, as $CF = 2b$ to $CE \sqrt{aa - cc - bb}$, ſo CE to CD the ſegment ſought, *n* 1.

PROBLEM X.

Commonly aſcribed to Archimedes.

THE *Diameter AB of a given ſemi-circle* (*Fig.* 22 *n.* 1.) *being any how divided in L, and from L erecting a Perpendicular LX, and upon the ſegments LA and LB having deſcribed two other ſemi-circles, whoſe ſemi-diameters are alſo given as well as that of the great Circle CB; to find the Radii FM and Vy of the little Circles that are to be ſo deſcribed, that they ſhall touch the Perpendicular LX, the Cavity of the greater ſemi-circle, and the Convexities of the leſ.*

SOLU-

SOLUTION.

I. For the Radius FM.

1. *Denomination.* Make CB $= a$, EB $= b$ then will CE $= a - b$; for which for brevities ſake put c. And let FM or FN or FK $= x$: Therefore EF will be $= b + x$, and CF (ſubſtracting FK from CK) $= a - x$. Wherefore now you'l have at leaſt the names of the three ſides in the △ CFE, ſo that according to Poblem 8. the Segment of the baſe GE may be determined (which indeed is determined already, as being $=$ LE—LG or MF *i. e.* $b - x$) for which in the mean time we will put y; and now will CG $= c - y$.

2. For the *Equation.* If the □ GE $= y$ be ſubſtracted from the □ EF $= bb + 2bx + xx$, you'l have the ſquare of the Perpendicular FG $= bb + 2bx + xx - yy$; and, if □ CG $= cc - 2cy + yy$ be ſubſtracted from the □ CF $= aa - 2ax + xx$, you'l have the ſame □ of the Perpendicular FG $= aa - 2ax + xx - cc + 2cy - yy$. Therefore $bb + 2bx + xx - yy = aa - 2ax + xx - cc + 2cy - yy$.

3. *Reduction.* And taking from both ſides the quantities xx and yy, $bb + 2bx = aa - 2ax - cc + 2cy$; and adding $2ax$ and cc, but taking away aa from both ſides, $bb + 2bx = aa - 2ax - cc + 2cy$; and adding $2ax$ and cc, and taking away from each ſide aa, $bb + 2bx + 2ax + cc - aa = 2cy$; and dividing by $2c$,

$$\frac{bb + 2bx + 2ax + cc - aa}{2c} = y.$$

but the ſame y or EG is $=$ EL—MF *i. e.* $b - x$. Therefore $\dfrac{bb + 2bx + 2ax + cc - aa}{2c} = b - x$;

which is a new and more principal Equation: And multiplying both ſides by $2c$ (you have a new Reduction) $bb + 2bx + 2ax + cc - aa = 2ac - 2bx$; and adding $2cx$, and tranſpoſing the others, $2bx + 2cx + 2ax = aa - bb - cc + 2bc$; and dividing by $2a + 2b + 2c$,

$$x = \frac{aa - bb - cc + 2bc}{2a + 2b + 2c}.$$

The Geometrical Conſtruction of this firſt Caſe. Add the determinate (*n.* 2.) quantity $2bc$ into one ſum with the quantity aa, as in the beginning, *n.* 3. Then from this ſum ſubſtract ſucceſſively the quantities bb and cc, and there will come out (the ſame *n.* 3.) FH, whoſe □ $= aa \dagger 2bc - bb - cc$: Which ſince it muſt be divided by $2a \dagger 2b \dagger 2c$, make (the ſame *n.* 3.) as $FI = 2a \dagger 2b \dagger 2c$ to $FH = \surd aa \dagger 2bc - bb - cc$, ſo FH to FM the Radius ſought of the little Circle to be deſcribed. This quantity FM being thus found, place it from L to G (*n.* 1.) and from G erect a Perpendicular, which being cut off at the interval CF (which may be had, if from CB or CK you cut off $FK = FM$) or from E at the interval EF (which is compoſed of the Radii EN and FN) gives the Center of the little Circle to be deſcribed.

The Arithmetical Rule. Add twice the ▭ CEB to the ſquare of the greateſt ſemi-diameter CB, and from the ſum ſubſtract the Aggregate of the □ □ CE and EB; divide the remainder by the ſum of all the three Diameters, (AB, AL and LB) *i. e.* by double the greateſt AB; and you'l have the Radius FM, *&c.* For Example ſake let a be $= 12$, $b = 4$; c will be $= 8$, and x will be produced $= 2\frac{2}{3}$.

II. *For the Radius Vy by help of the obtuſe-angled △ DVC.*

1. *Denomination.* $CA = a$ as above, DA or $DL = b$, and putting x again for the ſought Vy or VK, CV will be $= a - x$, DL or $DR = b$, and conſequently $DV = b \dagger x$, and $DC = a - b$, for which for brevity's ſake we will put c. Now you'l have at leaſt in Denomination in the △ CVD the three ſides, ſo that according to Problem 9. the ſegment CW may be determined, for which in the mean while we will put y; then will $DW = c \dagger y$, which is the ſame as DL—WL or Vy *i. e.* $b - x$.

2. For the *Equation.* If the □ $CW = yy$ ſubſtract it from the □ $CV = aa - 2ax \dagger xx$, and you'l have the □ of the Perpendicular VW, $aa - 2ax \dagger xx - yy$; and if the □ $DW = cc \dagger 2cy \dagger yy$, ſubſtract it from the □ $DV = bb \dagger 2bx \dagger xx$, and you'l have the ſame □ of the Perpendicular $VW = bb \dagger 2bx \dagger xx - cc - 2cy - yy$. Therefore

$$aa - 2ax \dagger xx - yy = bb \dagger 2bx \dagger xx - cc - 2cy - yy.$$

3. *Reduction.* Therefore taking from both ſides xx and yy, $aa - 2ax = bb + 2bx - cc - 2cy$; and adding $2cy$ and $2ax$, $aa + 2cy = bb + 2ax + 2bx - cc$; and ſubſtracting aa, $2cy = bb - aa + 2ax + 2bx - cc$, and dividing by $2c$,

$$y = \frac{bb - aa + 2ax + 2bx - cc}{2c}$$

But if you add to the ſame y or CW DC $= c$, you'l have

$$DW = \frac{bb - aa + 2ax + 2bx - cc + c}{2c} \quad i.\ e.$$

reducing this c to the ſame Denomination,

$$\frac{bb - aa + 2ax \ \ 2bx + cc}{2c} = DW.$$ But the ſame DW $= DL - WL = b - x$. Therefore

$$\frac{bb - aa + 2ax + 2bx + cc}{2c} = b - x,$$ and multiplying $2c$, $bb - aa + 2ax + 2bx + cc = 2bc + 2cx$, and adding $2cx$, and tranſpoſing the reſt, $2ax + 2bx + 2cx = aa - bb - cc + 2bc$, and dividing by $2a + 2b + 2c$,

$$x = \underline{aa - bb - cc + 2bc}$$ juſt as above in the firſt Caſe.

4. *The Geometrical Conſtruction* therefore will be the ſame as there. See *Fig.* 23. *n.* 4. and 5.

5. *The Arithmetical Rule* is alſo the ſame, but the given quantities in this Example, which the figure of the Problem will ſhew, thus vary, while a remains 12, b will be 8, and c 4, from which *data* (or given quantities) there will notwithſtanding come out again, for x or the Radius Vy $2\frac{2}{3}$.

II. *Some*

II. *Some Examples of ſimple or pure Quadratick Equations.*

PROBLEM I.

T*O make a Square equal to a given Rectangle*; i. e. *having given the ſides of the Rectangle, to find the ſide of an equal Square,* Eucl. Prop. 14. Lib. 2. Suppoſe *e. g.* the given ſides of the Oblong to be AB and BC (*Fig.* 24.) to find the Line BD whoſe ſquare ſhall be equal to that Rectangle.

SOLUTION.

Make AB $= a$ and BC $= b$, and the ſide of the ſquare ſought $= x$, and the Equation will be $ab = xx$; and extracting the root on both ſides $\surd ab = x$.

Geometrical Conſtruction. Join AB and BC in one right line, and deſcribing a ſemi-circle upon the whole AC, from the common juncture B erect the Perpendicular BD which will be the ſide of the ſquare ſought, according to Caſe 1. of the Effection of pure quadraticks.

Arithmetical Rule. Multiply the given ſides of the Oblong by one another, and the ſquare root extracted out of the Product will be the ſide of the ſquare ſought.

PROBLEM II.

T*HE ſquare of the Hypothenuſa in a right angled △ being given, as alſo the difference of the other two ſquares to find the ſides.* E. g. If the Hypothenuſa be BC (*Fig.* 25.) and the difference of the ſquares of both the legs, and conſequently its Leg alſo BE given (for the ſquares being given the ſides are alſo given geometrically) to find the ſides of the right-angled △ which ſhall have theſe conditions; or more plainly, to find one ſide *e. g.* the leſſer which being found, the other, or the greater, will be found alſo.

SOLUTION.

Let the □ of the given Hypothenuſa $= aa$, and the ſquare by which the two other differ $= bb$. Let the leſs ſide $= x$, and its □ $= xx$. Wherefore the greater will be $xx + bb$. And ſince the ſum of theſe is = to the □ of the Hypothenuſa, you'l have $2xx + bb = aa$; and ſubſtracting bb,

$2xx = aa - bb$; and dividing by 2

$xx = \frac{aa - bb}{2}$. Therefore

$$x = \sqrt{\frac{aa - bb}{2}}.$$

Geometrical Conſtruction. Having deſcribed a ſemi-circle on BC, and applyed therein BE, the □ EC will $= aa - bb$; and having deſcribed another ſemi circle upon EC divided into two Quadrants the □ DC will be $\frac{aa-bb}{2}$, and ſo DC $= \sqrt{\frac{aa-bb}{2}}$, or the ſide ſought; which being alſo transferr'd upon the other ſemi-circle deſcrib'd on BC, *viz.* from C to A gives the other ſide AB and the whole △ ſought.

The Arithmetical Rule. From the ſquare of the Hypothenuſa ſubſtract the given difference, and the ſquare root extracted out of half the remainder gives the leſſer ſide of the △ ſought.

PROBLEM III.

H*Aving an equilateral* △ *ABC given* (*Fig.* 26. *n.* 1.) *to find the Center and Semi-diameter of a Circle that ſhall circumſcribe it.* i. e. Find the BD the ſide of an Hexagon that may be inſcribed in it. For if we conſider the thing as already done; it will be manifeſt that the ſide of the Hexagon BD will fall perpendicularly on the ſide of the △ AB, as making an angle in a ſemi-circle, ſo having biſected the Hypothenuſa DA you'l have E the Center ſought.

SOLUTION.

Make the ſide of the Triangle $AB = a$, $BD = x$, then will $AD = 2x$. Since therefore the ſquare BD *i. e.* xx being ſubſtracted out of the ſquare AD *i. e.* $4xx$, there remains the ſquare AB $3xx$, you'l have the Equation

$$3xx = aa \text{; and dividing by } 3$$
$$xx = \frac{aa}{3} \text{; therefore}$$
$$x = \sqrt{\frac{aa}{3}}.$$

The Geometrical Conſtruction. Having produced AB (*n.*2.) to F a third part of it, the ſquare of a mean proportional BD between BF and BA will be $\frac{1}{3}aa$ or $\frac{aa}{3}$, and ſo the Line BD $= \sqrt{\frac{aa}{3}}$. Therefore the Hypothenuſa DA being divided in two in E, or at the interval BD, making the interſection from B and A, you'l have the Centre ſought.

The Arithmetical Rule. Divide the ſquare of the given ſide into three equal parts, and the ſquare Root of a third part will give the ſemi-diameter AE or BE ſought, by the interſection of two of which you'l have the Centre.

PROBLEM IV.

H*Aving given, in a right-angled Parallelogram, the Diagonal, or for a right-angled* △, *the Hypothenuſa and the proportion of the ſides, to find the ſides ſeparately and conſtruct the Parallelogram or* △. Suppoſe *e. g.* the given Diagonal to be AB (*Fig.* 27. *n.* 1.) and the given reaſon of the ſides as AD to DE, to find the ſides.

SOLUTION.

Make $AB = a$, the reaſon of AD to DE as b to c; mak the leſſer ſide $= x$, then will the greater be $\frac{cx}{b}$.

Fo

For the *Equation*, the □ □ of the ſides are $xx \dagger \frac{ccxx}{bb} = aa$, □ AB; and multiplying both ſides by *bb*,

$bbxx \dagger ccxx = aabb$; and dividing by $bb \dagger cc$,

$$xx = \frac{aabb.}{bb \dagger cc}$$ Therefore

$$x = \Upsilon\,\overline{\frac{aabb}{bb \dagger cc}}$$ i. e. Extracting the roots as far as poſſible

$$x = \frac{ab}{\Upsilon\,\overline{bb \dagger cc}}$$

Another Solution.

Call the name of the given reaſon *e*, ſo that aſſuming any line for unity, the value of *e* may alſo be expreſſed by a right line, which ſhall be equal *e. g.* to DE above. Wherefore becauſe we make the leſs ſide *x*, the greater will be *e x*, and ſo

$xx \dagger eexx = aa$, i. e. dividing by $1 \dagger ee$,

$$xx = \frac{aa}{1 \dagger ee}$$

$$x = \Upsilon\,\overline{\frac{aa}{1 \dagger ee}},$$ i. e.

$$x = \frac{a}{\Upsilon\,\overline{1 \dagger ee}}$$

The Geometrical Conſtruction. The laſt Equation above being reduc'd to this proportion as the $\Upsilon\, bb \dagger cc$ to *b* ſo *a* to *x*, make (*n* 2) AD and DE at right angles, and AE will be $= \Upsilon\, bb \dagger cc$, and continuing AE and AD make as AE to AD ſo AB to AC the leſſer ſide ſought. Having therefore drawn BC which determines the leſſer ſide AC, the greater ſide and ſo the △ ABC will be already formed, and may be eaſily compleated into a Rectangle. In the other Solution the laſt Equation agrees with the precedent (for it gives us this proportion as $\Upsilon\, 1 \dagger ee$ to 1 ſo *a* to *x* in which 1 is $= b$, and $ee = cc$ by what we have ſuppoſed) and ſo the Conſtruction will be the ſame.

The Arithmetical Rule may be more commodiouſly expreſſed by this laſt Equation under the laſt form but one, after this way, divide the □ of the Diagonal by the □ of the name of the Reaſon leſſen'd by unity, and the root extracted out of the Remainder is the leſſer ſide ſought.

PROBLEM V.

(Which is in *Pappus Alexandrinus*, and in *Cartes*'s Geometry, p. 83. in a Biquadratick affected Equation, and p. 84. he gives us thereon a very remarkable Note.)

Having given the Square AD (Fig. 28.) and a right line BN, you are to produce the ſide AC to E, ſo that EF drawn from E towards B ſhall be equal to BN.

It will be evident, if you imagine a ſemi-circle to paſs thro' the points B and E, that the moſt commodious way will be to find the line DG, that you may have the Diameter BG; upon which having afterwards deſcribed a ſemi circle, there will be need of no other operation to ſatisfie the queſtion, than to produce the ſide AC 'till it occur to the preſcrib'd Periphery.

SOLUTION.

(*As found by* Van Schooten, p. 316. *in his Comment on* Cartes's *Geometry, which we will here give ſomewhat more diſtinct.*)

1. *Denomination.* Make BD or DC = *a*, BN or FE = *c*, BF = *y*, and DG = *x*; the Perpendicular EH will be = *a*, and EG = BF, *viz. y* (becauſe the △ EHG is ſimilar to △ EDF, by *n.* 3 *Schol* 2. *Prop.* 34. *Lib.* 1. *Matheſ. Enucl.* and BD in the one = to EH in the other) and BG = *a* † *x*, BE = *y* † *c*; and BH will have its Denomination, if you make (by reaſon of the ſimilarity of the △ △ BFD and BEH)

as BF to BD ſo BE to BH
y — a — $y+c$ — $\frac{ay+ac}{y}$.

and you'l have alſo HG $= a + x - \frac{ay - ac}{y}$, i. e. having reduc'd them all to the ſame Denomination, $\frac{ay + xy - ay - ac}{y}$, i. e. $\frac{xy - ac}{y}$. Having therefore named all the lines you have occaſion for, you muſt find two Equations, becauſe there are aſſumed two unknown quantities, *viz.* x and y.

2. *For the firſt Equation and its Reduction.* By reaſon of the ſimiliarity of the △△ BGE and BEH,

as BG to GE ſo BE to EH
$a+x$ — y — $y+c$ — a: Therefore the Rectangle of the Extremes will be = to the Rectangle of the means, *i. e.* $yy + yc = aa + ax$, and taking from both ſides yc,
$yy = aa + vx - yc$.

3. *For the ſecond Equation and its Reduction.* Since
BH, HE and HG
$\frac{ay+ac}{y}$ — a — $\frac{xy-ac}{y}$ are continual proportionals, the Rectangle of the extreams are equal to the ſquare of the mean, *i. e.* $\frac{axyy + acxy - aacy - aacc}{yy} = aa$; and multiplying both ſides by yy, and dividing by a
$xyy + cxy - acy - acc = ayy$, and taking away ayy and tranſpoſing the reſt,
$xyy - ayy = acy - cxy + acc$; and dividing by $x - a$
$yy = \frac{acy - cxy + acc}{x - a}$; i. e. dividing actually as far as may be by $x - a$, $yy = - cy + \frac{acc}{x-a}$.

4. The compariſon of theſe two Equations thus reduc'd, gives a third new one, in which there will be only one unknown quantity, *viz.*
$- cy + aa + ax = - cy + \frac{acy}{x-a}$; and adding to both ſides cy,

$aa + ax = \frac{acc}{x-a}$; and multiplying by $x - a$,

$aax + axx - a^3 - aax = aac$; i. e. $axx - a^3 = acc$; and dividing both ſides by a, $xx - aa = cc$; and adding aa, $xx = aa + cc$. Therefore $x = \sqrt{aa + cc}$.

5. *The Geometrical Conſtruction*, which is the ſame *Pappus* preſcribes in *Cartes*, *viz.* having prolong'd the ſide of the ſquare BA to N, ſo that BN ſhall be = to a given right line, ſince BA is $= a$, and BN $= c$, the Hypothenuſa DN will be $= \sqrt{aa + cc} = x$. Having therefore made DG = DN, and deſcrib'd a ſemi-circle upon the whole line BG, if AC be prolonged until it occur to the Periphery in E, you'l have done that which was requir'd.

PROBLEM VI.

(Which *Van Schooten* has in his *Comment*, p. m. 150, and following.)

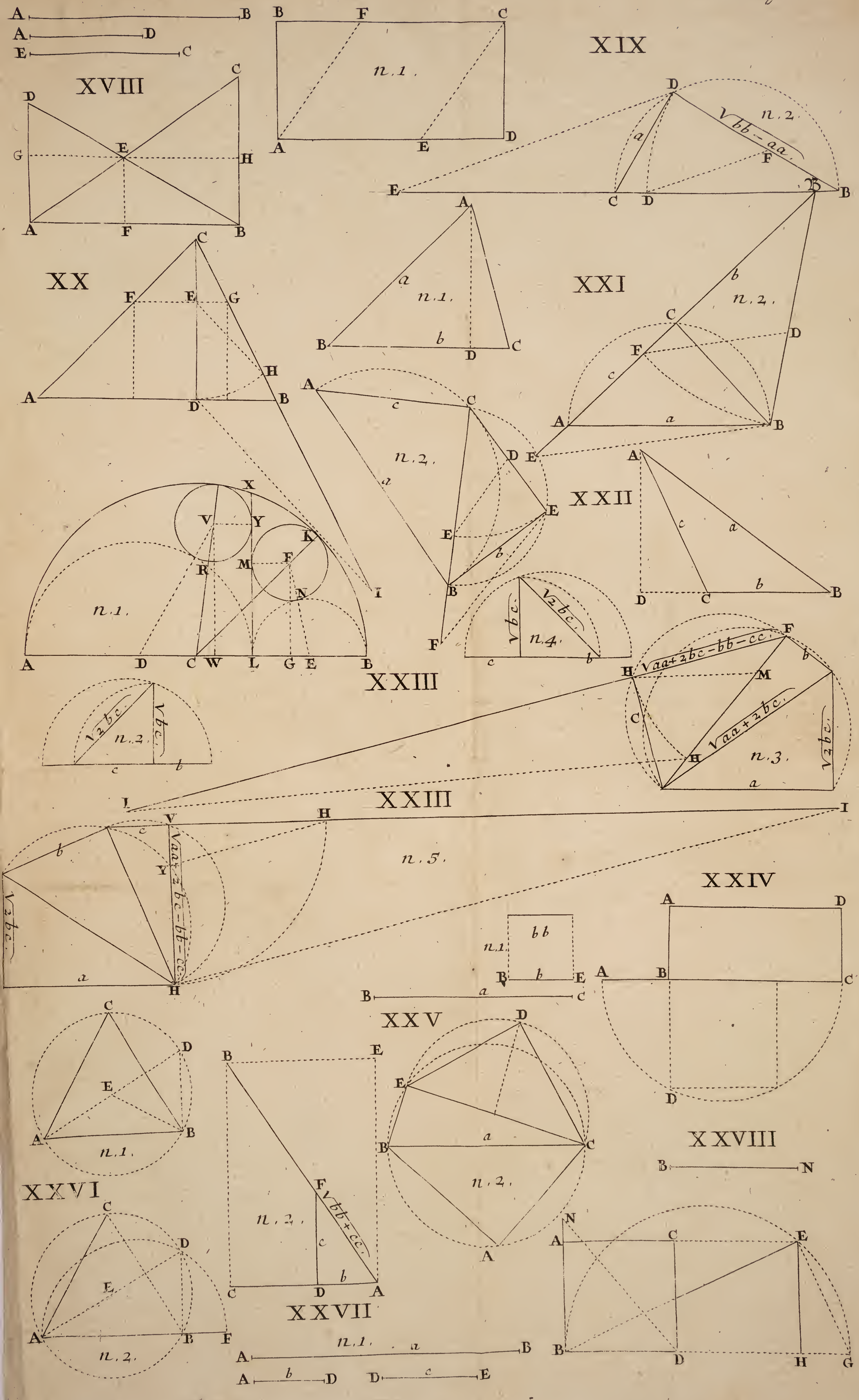
Having given a right line AB, from the ends of it A and B (*Fig.* 29.) *to inflect two right lines AC and BC, which ſhall contain an angle ACB = to the given one D, and whoſe ſquares ſhall be in a given proportion to the Triangle ACB*, viz. *as 4d to a.*

Viz. You muſt determine the point C, which the two right lines AH and HC or EH and HC will do, aſſuming the middle point E in the line AB. Wherefore here will be two unknown quantities HE and HC, and conſequently two Equations to be found in the Solution; one whereof the given proportion in the Queſtion ſupplies us with, and the other we have from the ſimilar Triangles AIC and GFD, which repreſent equal angles.

SOLUTION.

Denomination. Make AE, half AB $= a$, HE $= x$ and HC $= y$; therefore AH will be $= a - x$ and HB $= a + x$; whence the Denomination of the ſquares AC and BC is eaſily had; *viz.* the one $aa - 2ax + xx + yy$, and the other, $aa +$

2ax


XVIII
XIX
n. 1.
n. 2.
$\sqrt{bb-aa}$
XX
XXI
n. 1.
n. 2.
XXII
XXIII
n. 1.
n. 2.
$\sqrt{2bc}$
$\sqrt{bc}$
n. 3.
$\sqrt{aa+2bc-bb-cc}$
$\sqrt{aa+2bc}$
$\sqrt{2bc}$
n. 4.
$\sqrt{2bc}$
$\sqrt{bc}$
XXIII
n. 5.
$\sqrt{aa+2bc-bb-cc}$
$\sqrt{2bc}$
XXIV
n. 1.
bb
XXV
n. 1.
n. 2.
XXVI
n. 1.
n. 2.
XXVII
n. 1.
n. 2.
$\sqrt{bb+cc}$
XXVIII

b

zad

$2ax + xx + yy$, ſo that the ſum of the ſquares is $2aa + 2xx + 2yy$. And the △ ACB will be = ay: And ſince the △ △ GFD and AIC are ſimilar, and the ſides of the former FD and FO arbitrary, ſo that for FD we may put b and for FC, c; and the ſides of the latter are determined by the ſimilitude of the △ △ ABI and HDB, as being right-angled ones, and having the common angle B; they will be obtain'd by making
as the Hypothen. BC to the Hypoth. AB ſo the baſe HB to the baſe BI, *i. e.*

$$e \;\text{—}\; 2a \;\text{—}\; a + x \;\text{—}\; \frac{2aa + 2ax}{e},$$

from whence ſubſtracting BC = e, there remains CI = $\frac{2aa + 2ax - ee}{e}$.

2. *For the firſt Equation,* by virtue of the Problem as $4d$ to a, ſo $2aa + 2xx + 2yy$ to ay. And the Rectangle of the extremes is = to the Rectangle of the means, *i. e.* $4ady = 2a^3 + 2axx + 2ayy$.

3. *For the other Equation,* ſince
as DF to FG ſo CI to AI

$$b \;\text{—}\; c \;\text{—}\; \frac{2aa + 2ax - ee}{e} \;\text{—}\; \frac{2ay}{e}$$

and the Rectangle of the extreams will again be = to the Rectangle of the means, i. e. $\frac{2ayb}{e} = \frac{2aac + 2acx - cee}{e}$;

and multiplying both ſides by e, $2ayb = 2aac + 2acx - cee$; which is the ſecond Equation.

4. *The Reduction of both Equations.*

The firſt was $4ady = a^3 + 2axx + 2ayy$.
Therefore dividing by $2a$, $2dy = aa + xx + yy$.
And ſubſtracting $aa + yy$, $2dy - aa - yy = xx$.

The latter Equation was $2ayb = 2aac + 2acx - eec$, i. e. ſubſtituting again the value ee, which was
$aa + 2ax + xx + yy$.

$$2ayb = 2aac + 2acx - aac - 2acx - cxx - cyy,$$

i. e. $2ayb = aac - cxx - cyy$; and by tranſpoſition,
$cxx = aac - 2ayb - cyy$; and dividing by c,
$xx = \frac{aa - 2ay - yy}{c}$, or $aa - yy - \frac{2ayb}{c}$,

or (putting $2f$ for $\frac{2ab}{c}$) $xx = aa - yy - 2fy$.

Wherefore we have the value of xx twice expreſſed, but by quantities partly unknown, becauſe y is found on both ſides. Wherefore now we muſt make a new compariſon of their values, whence you'l have this new

5. *Third Equation*, in which there is only one of the unknown quantities:

$2dy - aa - yy = aa - yy - 2fy$; and adding on both ſides both yy and aa,

$2dy = 2aa - 2fy$; or dividing by 2,

$dy = aa - fy$; and tranſpoſing fy,

$dy + fy = aa$; and dividing by $d + f$,

$y = \frac{aa}{d+f}$; which is the value of the quantity y in known terms.

But this value in one of the precedent Equations, *viz.* in this $xx = 2dy - aa - yy$, being ſubſtituted for y and its ſquare for yy, will give

$$xx = \frac{2daa}{d+f} - aa - \frac{a^4}{dd+2df+ff};$$ i. e. all being reduced to the ſame denomination,

$$xx = \frac{aadd - aaff - a^4}{dd+2df+ff};$$ and

$$x = \sqrt{\frac{aadd - aaff - a^4}{dd+2df+ff}}.$$

6. *The Geometrical Conſtruction*, which *Schooten* gives us p. 153. Having made the angle KAB (*n.* 2. *Fig.* 29.) equal to the given one D, erect from A, AL perpendicular to KA, meeting the Perpendicular EM in L; and from the Centre L, at the interval of the given right line d, deſcribe a Circle that ſhall cut KA and EL produced to K and M. Then aſſuming EN = KA, join MA, and from N draw NH parallel to it, which ſhall meet AB in H. Afterwards, having deſcribed from L, at the interval LA, the ſegment of a Circle ACB, draw from H, HC perpendicular to AB meeting the circumference in C, and join AC, CB.

NB. The reaſon of this elegant Conſtruction, which the Author conceal'd, for the ſake of Learners we will here ſhew: 1. Therefore, he reduc'd the laſt Equation (extracting the root, as well as it could bear, both of Numerator and *Denominator*) to this: $x = \frac{a}{d+f}$ multipl. by $\sqrt{dd-ff-aa}$, ſo that after this way the Conſtruction would be reduc'd to this proportion, as $d+f$ to a ſo $\sqrt{dd-ff-aa}$ to x. 2. He made the angle KAE $=$ to the given one D, and the angle KAL a right one, ſo that having deſcribed the ſegment of a Circle from L the inſcribed angle will alſo be made equal to the given one, according to the 33. *Lib.* 3: *Eucl.* 3. By doing this, EL expreſſes the quantity f, ſince by reaſon of the ſimilarity of the △ △ KOA ſ. GFD, *n.* 1. and AEL (for the angles LAE and AKO are equal, becauſe each makes a right one with the ſame third Angle KAO) you have

as KO to OA ſo AE to EL i. e. f

$$c \text{ —— } b \text{ —— } a \text{ —— } \frac{ab}{c}$$

4. Making now LM and LK $= d$ you had EM $= d+f$, and AK $= \sqrt{dd-ff-aa}$ (for the □ AL is $=$ to $aa+ff$, which being ſubſtracted from □ LK $= dd$, there remains □ AK $= dd-aa-ff$.)

5. Wherefore there now remains nothing to conſtruct the laſt Equation above, but to make EN $=$ AK, and to draw HN parallel to AM; for thus was the whole proportion

as EM to EA ſo EN to EH

$$d+f \text{ —— } a \text{ —— } \sqrt{dd-ff-aa} \text{ to } x. \text{ Q. e. f.}$$

For the point H being determined, a perpendicular HC thence erected in the ſegment already deſcribed defines the Point C, which anſwers the Queſtion.

PROBLEM VII.

Having given the four ſides of a Quadrangle to be inſcribed in a Circle, to find the Diagonals and their Segments, and ſo to conſtruct the Quadrangle, and inſcribe it in the Circle. As *e. g.* ſuppoſe the given ſides are AB, BC, CD, DA (*Fig.* 30 *n.* 1.) which now we ſuppoſe to be joined into a quadrangle inſcrib'd in the Circle, the Diagonals alſo AC and BD being drawn (*n.* 3.) to find firſt the ſegments of the diagonals A*e*, B*e*, *&c.* which being had, the Conſtruction is ready.

SOLUTION.

Denomination. Make AB $= a$, AC $= b$, CD $= c$, DA $= d$, A*e* $= x$ [for this ſegment alone being found, the reſt will be found alſo, as will be evident from the proceſs.] Since therefore the vertical angles at *e* are equal, and likewiſe the angles in the ſame ſegment BCA, BDA, alſo DAC, DBC, *&c.* are equal, the Triangles A*e*D and B*e*C, alſo A*e*B and C*e*D are ſimilar: wherefore it will follow that,

1. As DA to A*e* ſo CB to B*e*
   d —— x —— b —— $\frac{bx}{d}$
2. As AB to B*e* ſo CD to C*e*
   a —— $\frac{bx}{d}$ —— c —— $\frac{bcx}{ad}$
3. As AB to A*e* ſo CD to D*e*
   a —— x —— c —— $\frac{cx}{a}$

Therefore the whole Diagonal AC will be $= x + \frac{bcx}{ad}$ and BD $= \frac{bx}{d} + \frac{cx}{a}$.

2. *The Equation.* But now by *Prop.* 48. *Lib.* 1. *Math. Enucl.* the Rectangle of the Diagonals is equal to the two Rectangles of the oppoſite ſides.

Diag. AC, $x + \frac{bcx}{ad}$

Diag. BD, $\frac{bx}{d} + \frac{cx}{a}$

$$\frac{cxx}{a} + \frac{bbcxx}{add}$$

Therefore the $\square$ of the Diagonals $\frac{bxx}{d} + \frac{cxx}{a} + \frac{bbcxx}{add} + \frac{bccxx}{aad}$ is $= ac + bd$.

3. *Reduction*, i. e. taking (*a*) for unity

$$\frac{bxx}{d} + cxx + \frac{bbcxx}{dd} + \frac{bccxx}{d} = c + bd;$$

i. e. the quantities on the left hand being reduc'd to the ſame denomination.

$$\frac{bdxx + cddxx + bbcxx + bccdxx}{dd} = c + bd;$$

and multiplying both ſides by *dd*,

$$bdxx + cdxx + bbcxx + bccdxx = cdd + bd^3;$$

and dividing both ſides by *d*

$$bxx + cxx + \frac{bbcxx}{d} + bccxx = cd + bdd;$$

and then dividing both ſides by $b + cd + \frac{bbc}{d} + bcc$,

$$xx = \frac{cd + bdd}{b + cd + \frac{bbc}{d} + bcc};$$

i. e. in the preſent caſe, where *b* by chance happens to be $= a$,

$$xx = \frac{cd + dd}{1 + cd + \frac{c}{d} + cc}$$

Therefore $x = \sqrt{\frac{cd + bdd}{b + cd + \frac{bbc}{d} + bcc}}$

or in our caſe

$$x = \sqrt{\frac{cd + dd}{1 + cd + \frac{c}{d} + cc}}$$

4. *The Geometrical Conſtruction*, which, by ſuppoſing *a* (and in the preſent caſe alſo *b*) to be unity, ought to deter-

mine, 1. The quantities cd, $\frac{c}{d}$, cc, and their aggregate with unity. 2. The aggregate of cd and dd. 3. To divide the one by the other. And, 4. To extract the root out of the quotient, or alſo to extract the roots firſt out of each quantity, and divide them by one another; which may all of them be ſeparately done in ſo many ſeparate Diagrams, but more elegantly connected together after the following or ſome ſuch like way. 1. Join AD and DC (*n.* 2.) into one line, and having deſcribed a ſemi-circle thereupon, erect the Perpendicular DE; and the line AE drawn will $= \sqrt{cd + dd}$. 2. Making the angle CAG at pleaſure, make AF $=$ AB, and draw CG parallel to the line DF, ſo FG will be $= \frac{c}{d}$. Now if, 3. in the vertical angle you make AH $=$ CD, the line HI drawn parallel to DF will cut off AI $= cd$. 4. In AK erected $=$ to AB, if you take AL $=$ AH or CD, and draw LM parallel to KH, you'l have AM $= cc$. 5. Having prolonged AG to N and AH to O, ſo that GN ſhall be $=$ AI + AM and AO $=$ AB or AK, and having deſcribed a ſemi-circle upon the whole line NO, a perpendicular erected AP will be $= \sqrt{1 + cd + \frac{c}{d} + cc}$; and ſo, 6. if AQ be made $=$ AE and AR $=$ AF or AB, and you draw a line RS from R parallel to PQ; AS will be $= x$, *i. e* the ſegment ſought Ae of the Diagonal AC; which being given, by force of the firſt Inference premis'd in the Denomination above, by drawing DS and, having made DT $=$ BC, TV parallel to it; you'l have alſo the other ſegment Be $=$ SV and by their Interſection on the line AB (*n.* 3.) the point e, thro' which the Diagonals muſt be drawn which will be terminated by the other given ſides, and thence you'l have the quadrilateral figure ABCD ſought, to be circumſcribed about the Circle, according to *Conſect.* 6 *Defin.* 8. *Matheſ. Enucl.*

NB. Unleſs we had here conſulted the Learner's caſe, the artifice of this Conſtruction might be propoſed after a more ſhort and occult way, thus: Make DE a mean proportional between AD and DC, and draw AE. Then having made any angle CAG, make AF $=$ AB, and at this Interval deſcribe the circle FROK, and draw CG parallel to DF. Moreover in the oppoſite vertical angle, having made $AH = CD$, draw HI parallel to DF, and having erected the perpendicular AK, and thence the abſciſſa $AL = AH$, make LM parallel to HK, and thence having prolonged AH to O, and GN being made equal to $AI + AM$, make AP a meam proportional between AO and AN, cutting the hidden circle in R; and laſtly having made $AQ = AE$, if RS be drawn parallel to QP, you'l have AS the value of x ſought, *&c.*

III. *Some Examples of Affected Quadratick Equations.*

PROBLEM I.

Having given, to make a right angled Triangle ABC, the differences of the leſſer and greater ſide, and of the greater, and the Hypothenuſa, to find the ſides ſeparately and form the Triangle. E. g. Having given the right line DB (*Fig.* 31.) for the difference of the perpendicular and baſe, and CE for the difference of the baſe and Hypothenuſa, to find the perpendicular AC, which being found, you'l have alſo, by what we have ſuppoſed, the baſe AB, and the hypothenuſa BC.

SOLUTION.

Make the difference $DB = a$, $CE = b$; put x for the perpendicular; the baſe, which is greater than that will be $x + a$ and the Hypothenuſa $x + a + b$. Therefore by vertue of the Pythagorick Theorem,

$$2xx + 2ax + aa = xx + 2ax + 2bx + aa + 2ab + bb;$$

and ſubſtracting from both ſides $xx + 2ax + aa$,

$$xx = 2bx + 2ab + bb.$$

Wherefore by the firſt caſe of affected quadratick Equations

$$x = b + \sqrt{2ab + 2bb}.$$

Conſtruction. Find a mean proportional AK between $AH = 2b$ and $AI = a$ (*n.* 2.) (*Fig.* 31.) and having made both AF and $AG = b$, place the Hypothenuſa KF from AL, and cut off GC equal to the hypothenuſa GL; thus you'l have AC the perpendicular of the Triangle ſought, and adding DB you'l alſo have the baſe AB, and from thence having drawn the hypothenuſa BC, it will be found to differ by the exceſs required CE.

The Arithmetical Rule. Join twice the product of the differences multiplyed by one another, to twice the ſquare of the difference of the baſe and the hypothenuſa; and if the ſquare root of this ſum being extracted be added to the aforeſaid difference, you'l have the perpendicular ſought. Suppoſe *e. g.* both the differences of CE and DB $= 10$.

PROBLEM II.

IN a right-angled △ having given the Hypothenuſa and ſum of the ſides, to find the ſides. E. g. If the Hypothenuſa BC be given (*Fig.* 32.) and the ſum of the ſides CAB, to find the ſides AB and AC ſeparately, to form the Triangle.

SOLUTION.

Make the Hypothenuſa BC $= a$, the ſum of the ſides $= b$. Make one ſide *e.g.* AB $= x$, then will the other ſide AC be $= b - x$. Therefore

$2xx - 2bx + bb = aa$; and adding $2bx$, and taking away bb, $2xx = 2bx + aa - bb$; and dividing by 2,

$$xx = bx + \frac{aa - bb}{2}.$$

Therefore according to Caſe 1. of affected Quadraticks,

$$x = \tfrac{1}{2}b + \sqrt{\tfrac{1}{4}bb + \frac{aa - bb}{2}}$$

$$\text{i. e. } \tfrac{1}{2}b + \sqrt{\tfrac{1}{2}aa - \tfrac{1}{4}bb}$$

$$\text{or } \tfrac{1}{2}b - \sqrt{\tfrac{1}{2}aa - \tfrac{1}{4}bb}.$$

The Geometrical Conſtruction. Having deſcribed a ſemi-circle upon BD $=$ BC ſo a apply therein the equal lines BE and DE, and having deſcribed another ſemi circle on BE apply therein BF $= \tfrac{1}{2}b$, to be prolonged farther out. Laſtly, if another little ſemi circle be deſcribed at the Interval EF, the whole line AB will be the true root or the ſide ſought, and GB the falſe root, *&c.*

The Arithmetical Rule. From half the ſquare of the Hypothenuſa ſubſtract the fourth part of the ſquare of the given ſum, and the root extracted out of the remainder, if it be added to half the ſum, will give one ſide of the Triangle; and ſubſtracted from the given ſum, will give alſo the other Suppoſe *e. g.* BC to be 20, and the ſum of the ſides 28.

PROBLEM III.

Having given again in the ſame △ the Hypothenuſa, as above, and the difference of the ſides DB (*Fig.* 33.) *to find the ſides.*

SOLUTION.

Make the leſs ſide x, the difference of the ſides $= b$; the greater ſide will be $x + b$. Let the Hypothenuſa be $= a$. Therefore, $2xx + 2bx + bb = aa$; and taking away $2bx + bb$,

$2xx = aa - 2bx - bb$; and dividing by 2,

$$xx = -bx + \frac{aa - bb}{2}.$$

Therefore by caſe 2, $x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{aa - bb}{2}}$

i. e. $-\frac{1}{2}b + \sqrt{\frac{1}{2}aa - \frac{1}{4}bb}.$

The Geometrical Conſtruction. Having deſcribed a ſemi-circle upon BD = BC or a, apply therein the equal lines BC and DC, and having deſcribed another ſemi-circle on DC apply in it DF $= \frac{1}{2}b$, and if at the ſame interval you cut off FA from FC, the remainder AC will be the leſſer ſide ſought, *&c.*

The Arithmetical Rule. From half the ſquare of the Hypothenuſa ſubſtract the ſquare of half the difference, and if you take half the difference from the root extracted out of the remainder, you'l have the leſſer ſide of the Triangle required, and by adding to it the given difference you'l have alſo the greater. *E. g.* Let the Hypothenuſa be 20, and the difference of the ſides 4.

PRO-

PROBLEM IV.

H*Aving given the Area of a right-angled Parallelogram, and the difference of the ſides to find the ſides.* E. g. If the Area is = to the ſquare of the given line DF, and the difference of the ſides ED (*Fig.* 34.) to find the ſides of the rectangle.

SOLUTION.

Make the given Area $= aa$, the difference of the ſides $= b$, the leſſer ſide x; then the greater will be $x + b$. Therefore the Area $xx + bx = aa$; and ſubſtracting bx

$$xx = -bx + aa.$$

Therefore according to caſe 2, $x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + aa}$.

The Geometrical Conſtruction. Join at right angles AG $= a$, and GH $= \frac{1}{2}b$, and having drawn AH and prolong'd it, deſcribe the little Circle at the interval GH: ſo you'l have AE the leſſer ſide, and AD the greater of the Rectangle ſought, *&c.*

The Arithmetical Rule. Add the given Area and the ſquare of half the difference, and having the ſum, ſubſtract and add the difference from or to the root extracted, and ſo you'l have the greater and leſs ſides of the rectangle.

PROBLEM V.

H*Aving given for a right-angled Triangle the difference of both the Legs from the Hypothenuſa, to find the ſides and ſo the whole Triangle.* E. g. Suppoſe the difference of the leſs ſide to be BD and of the greater DE (*n.* 1. *Fig.* 35.) to find the ſides themſelves, and ſo make the Triangle.

SOLUTION.

For BD put a, for DE, b. Let the greater ſide be x; the Hypothenuſa will be $x + b$; therefore the leſſer ſide will be $x + b - a$. Now the □ □ of the ſides are = to the □ of the Hypothenuſa, *i. e.* $2xx - 2ax + 2bx + bb - 2ab + aa = xx$

$= xx + 2bx + bb$; and taking away $xx + 2bx + bb$,

$$xx - 2ax - 2ab + aa = 0$$; and adding $2ax$ and $2ab$, and taking away aa,

$$xx = 2ax + 2ab - aa.$$ Therefore

$$x = a + \sqrt{aa + 2ab - aa}$$; i. e.

$$x = a + \sqrt{2ab}.$$

The Geometrical Conſtruction. Between the given differences BD and DE find (*n.* 2.) a mean proportional DF, and join to it at right angles the equal line FG, and cut off DH equal to DG; and ſo you'l have BH the greater ſide of the triangle ſought. This being prolong'd to C, ſo that HC ſhall be $= b$, having deſcribed a ſemi-circle upon the whole line BC apply therein BA = BH; and having drawn AC, the Triangle ſought ABC, will be formed.

The Arithmetical Rule. If the ſquare root extracted from the double rectangle of the differences be added to the greater difference, you'l have the greater ſide ſought, *&c.*

PROBLEM VI.

HAving given, *to make two unequal Rectangles, but of equal heighth, the ſum of their Baſes with the Area of either* (viz. *the greater,*) *and the proportion of the ſides of the other* (viz. *the leaſt,*) *to find the ſides ſeparately.* E g. Let the ſum of the baſes be AB (*n.* 1. *Fig.* 36.) and the ſquare of the line BC = to the Area of the greater rectangle; and let the ſides of the leſſer rectangle be to one another as CD to DE: To find the ſides of both the rectangles; *i. e* to find the common altitude, which being found the other ſides will be eaſily obtain'd from the *Data*; or to find the baſe of the greater which, with the ſame eaſe, will diſcover the reſt.

SOLUTION.

Make AB $= a$, and the Area of the greater rectangle $= bb$; and the proportion of the altitude to the baſe in the leſſer, as c to d; to find *e. g.* the greater baſe which call x. Therefore the common altitude will be $= \frac{bb}{x}$, and the baſe of the leſſer rectangle $= a - x$.

Wherefore you'l have for the Equation,

as c to d ſo $\frac{bb}{x}$ to $a - x$.

Therefore $ac - cx = \frac{bbd}{x}$; and multipl. by x,

$acx - cxx = bbd$; and adding cxx and taking away bbd, $acx - bbd = cxx$. Now that you may conveniently divide both ſides by c, make firſt as c to b ſo d to a fourth which call f, and then put cf for bd, and you'l have

$acx - bcf = cxx$; and dividing by c

$ax - bf = xx$; and ſo according to caſe 3.

$$x = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bf}$$

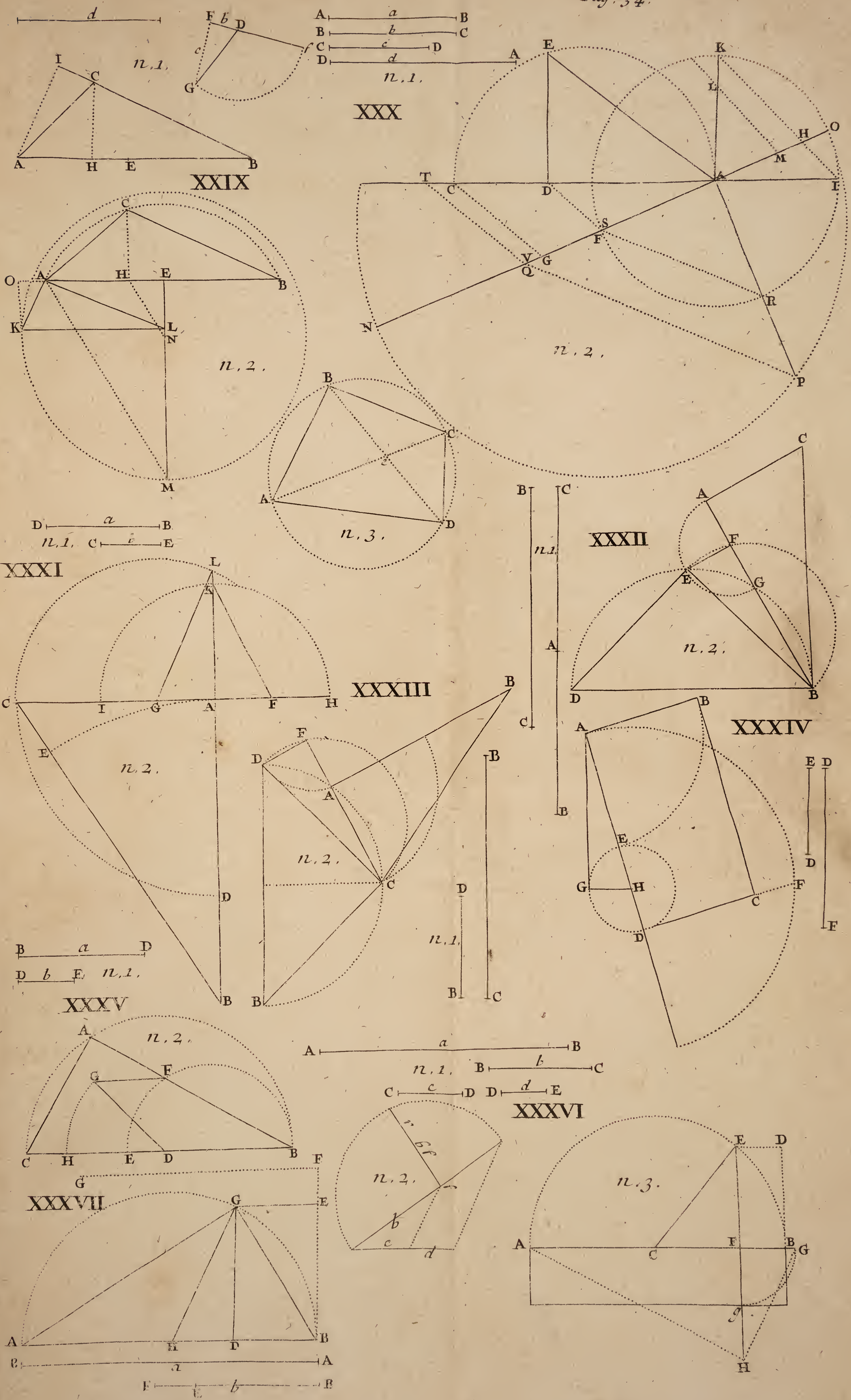
$$\text{or } \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bf}.$$

The Geometrical Conſtruction. Find firſt the quantity f (*num.* 2. *Fig.* 36.) according to the following proportion, as c to b ſo d to f; and a mean proportional between b and f will be $= \sqrt{bf}$. Then having at the interval $\frac{1}{2}a$ deſcribed a ſemi circle (*n.* 3.) upon the given line AB, and erected $\mathrm{BD} = \sqrt{bf}$, and having made EF = to it, CF will be $\sqrt{\tfrac{1}{4}aa - bf}$: To which AC being added will give x for one value and FB for the other. And for the common altitude, which we called $\frac{bb}{x}$, make as x to b, ſo b to a fourth, *i. e.* as AF to FH ſo FH to FG; which will be the altitude of both rectangles Ag and Bg which may now eaſily be conſtructed.

The Arithmetical Rule might eaſily be had from this Equation reduced; but you may have it more commodiouſly from this other

SOLUTION.

Let the Denomination remain the ſame as above, only here put x for the common altitude, and expreſs the reaſon of the leſſer baſe of the rectangle to this altitude by e, and that baſe will be $= ex$: Therefore the baſe of the greater Rectangle will be $= a - ex$. Having now multiplyed the common altitude by each baſe, the area of the greater rectangle will be $ax - exx$, and hence you'l have the Equation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$ax - exx = bb$; and adding exx, and taking away bb, $ax - bb = exx$; and dividing by e, $\frac{ax - bb}{e} = xx$. Therefore by caſe 3.

$$x = \frac{a}{2e} + \sqrt{\frac{aa}{4ee} - \frac{bb}{e}} \text{ or } x = \frac{a}{2e} - \sqrt{\frac{aa}{4ee} - \frac{bb}{e}}.$$

Wherefore now this will be the *Arithmetical Rule.* If from the fourth part of the ſquare of the ſum of the baſes divided by the □ of the name of the reaſon you ſubſtract the given area divided by the ſame name of the reaſon, and if the root extracted out of the remainder be added to or ſubſtracted from half the ſum of the baſes divided by the ſame name of the reaſon; this ſum or remainder will give the altitude of the given Rectangles, and that multiplyed by the name of the reaſon one of the baſes: And that being ſubſtracted from the given ſum of the baſes will give the other baſe. For Example, let the ſum of the baſes be 16, the area of one of the rectangles 30, the name of the reaſon which the common altitude has to the baſe of the other rectangle = 2. There will come out the common altitude, on the one ſide 5, on the other 3, *&c.*

PROBLEM VII.

Having given the Perpendicular of a right-angled Triangle let fall from the right angle, and its Baſe, to find the ſegments of the Baſe, and ſo to form the Triangle.

E g. if the baſe of the right-angled Triangle you are to form be AB (*Fig.* 37.) and the length of the perpendicular BE or BF; to find the ſegments of the baſe, and ſo the point D, from which you are to make the perpendicular CD, to form the Tiangle ABG.

SOLUTION.

Let the given baſe be $= a$; and the given perpendicular $= b$: Then will one of the ſegments of the baſe be $= x$; and the other $= a - x$, and b a mean proportional between the ſaid ſegments, *i. e.*

x to b as b to $a - x$;

Therefore $ax - xx = bb$; and by adding xx and taking away bb,

$ax - bb = xx$. Therefore by caſe 3.

$$x = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bb}$$

$$\text{or } \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bb}.$$

The Geometrical Conſtruction. Having deſcribed a ſemicircle upon the given line AB, if you erect the perpendicular BE, and from the point G (which is determined by EG parallel to AB) let fall GD equal to it, you will have the two ſegments ſought, AD $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$ and DB $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$, which Conſtruction, it cannot be denyed, but it may be evident to any attentive perſon even without the Analyſis.

But that caſe may by the by be taken notice of wherein the given perpendicular would not be BE but BF. For in this caſe the perpendicular BF being erected upon AB, the parallel FG would not cut the ſemi-circle; which is an infallible ſign that the Problem in this caſe is impoſſible, where the perpendicular is ſuppoſed to be greater than half the baſe; which is inconſiſtent with a right angle.

The Arithmetical Rule. From the ſquare of half the baſe take the ſquare of the given perpendicular, and add or ſubſtract the ſquare root extracted out of the remainder, to or from half the baſe; and on the one hand the ſum will give the greater ſegment, and on the other the difference will give the leſs.

PROBLEM VIII.

H*Aving given the perpendicular of a right-angled Triangle that is to be let fall from the right angle, and the difference of the ſegments of the baſe, to find the ſegments, and deſcribe the Triangle.*

E. g. If the perpendicular is, as above, BE, and the difference of the ſegments AH (*n.* 1. *Fig.* 38.) to find the ſegments AD and DB, from whoſe common term you are to erect a perpendicular DG or DC to form the Triangle.

SOLUTION.

Make the leſſer ſegment $= x$, and the difference of the ſegments $= a$, the greater ſegment will be $x + a$. Make the given perpendicular as before $= b$: Therefore you'l have

as $x + a$ to b ſo b to x; and conſequently,
$xx + ax = bb$; and ſubſtracting ax,
$xx = bb - ax$. Wherefore according to caſe 2.

$$x = -\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}.$$

The Geometrical Conſtruction. Make HD $= \frac{1}{2}a$, DG equal and perpendicular to b; HG will be $= \sqrt{\frac{1}{4}aa + bb} =$ HB or HA, *viz.* having drawn a ſemi-circle from H thro' G. Therefore DB is the leſs ſegment, and AD the greater; and having drawn AG and BG, or on the other ſide, (making the perpendicular DC $=$ DG) having drawn AC and BC, the Triangle will be conſtructed. Or with *Cartes*, make (*n.* 2.) HE $= \frac{1}{2}a$ and EB $= b$, and having deſcribed a Circle from H thro' E draw BHA; and ſo you'l have the two ſegments ſought AB the greater and DB the leſſer.

The Arithmetical Rule. Join the ſquares of the half difference and perpendicular into one ſum, and then having extracted the root ſubſtract half the difference from it; and the remainder will be the leſſer ſegment ſought; and having added the difference you'l have alſo the greater.

PROBLEM IX.

Having given for a right-angled Triangle one ſegment of the baſe and the ſide adjacent to the other ſegment, to find the reſt and conſtruct the Triangle.

As if the leſſer ſegment of the baſe DB be given (*Fig.* 39. *n.* 1.) and the ſide AC adjacent to the other ſegment; to find the greater ſegment of the baſe, which being found the reſt are eaſily obtain'd, and conſequently the whole Triangle.

SOLUTION.

Make the greater ſegment $= b$, the given ſide $= c$, the ſegment ſought $= x$. Now if we ſuppoſe the triangle ABC to be already found, it is evident, 1. If from the ſquare of AC you ſubſtract the □ AD, you'l have the □ CD $= cc - xx$. 2. The ſame □ CD may alſo be otherwiſe hence obtain'd, becauſe, the angle at C being a right one, CD is a mean proportional between BD and DA, *i. e.* between b and x; whence the rectangle of the extremes bx is $=$ □ of the mean CD. Wherefore now it follows, 3. that

$cc - xx = bx$; and adding xx,
$cc = bx + xx$; and ſubſtracting bx,
$-bx + cc = xx$. Therefore according to caſe 2.
$x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + cc}$.

Geometrical Conſtruction. Join EF $= \frac{1}{2}b$ (*n.* 2. *Fig.* 39.) and FA $= c$ at right angles, and having deſcribed a Circle from E thro' F draw AEB; ſo you'l have DA the greater ſegment and DB the leſs; having erected therefore a perpendicular from D, and deſcribed a ſemi-circle upon AB, you'l have C the vertex of the triangle ſought, whence you are to draw the ſides AC and BC.

The Arithmetical Rule. Join the □ of half the given ſegment, and the □ of the given ſide into one ſum; and having extracted the root of it, if you thence take half the given ſegment, you'l have the ſegment ſought.

PROBLEM X.

Having given in an oblique angled Triangle the perpendicular height, and the difference of the ſegments of the baſe, and the difference of the other ſides, to find the ſides and form the triangle.

As, if the altitude CD be given (*n.* 1. *Fig.* 40.) and alſo the difference of the ſegments of the baſe EB, and the difference of the ſides FB (as is evident from the triangle ABC (*n.* 2.) conceived to be ſo formed beforehand) and you are to determine the baſe it ſelf and both ſides, *&c.*

SOLUTION.

Make the given perpendicular CD $= a$ (ſee *n.* 2. *Fig* 40.) EB $= b$, FB $= c$: For the leſſer ſegment of the baſe AD put x, and the greater will be $x + b$. It is now evident that you may obtain the □ CB by the addition of the □ □ DC and BD, $aa + xx + 2bx + bb$; and the □ AC by the addition of the □ □ AD and DC, *viz.* $aa + xx$: So that the ſide AC will be $= \sqrt{aa + xx}$, and the ſide BC $= \sqrt{aa + xx + 2bx + bb}$. But ſince alſo this ſame ſide BC may be obtain'd by adding the difference c to the ſide AC, ſo that it ſhall be $= c + \sqrt{aa + xx}$: you'l have this Equation,

$c + \sqrt{aa + xx} = \sqrt{aa + xx + 2bx + bb}$; and ſquaring both ſides,

$cc + aa + xx + \sqrt{4ccaa + 4ccxx} = aa + xx + 2bx + bb$; and ſubſtracting from both ſides $cc + aa + xx$,

$\sqrt{4ccaa + 4ccxx} = 2bx + bb - cc$;
and again ſquaring it,

$4ccaa + 4ccxx = 4bbxx + 4b^3x - 4bcx + b^4 - 2bbcc + c^4$;
and ſubſtracting from both ſides $4ccxx$ (becauſe c is leſs than b) and tranſpoſing the reſt,

$$\begin{array}{l} 4ccaa - 4b^3x - b^4 \\ \quad + 4bccx + 2bbcc = 4bbxx - 4ccxx; \\ \qquad - c^4 \end{array}$$

and dividing by $4bb - 4cc$,

$$\frac{\begin{array}{l} 4cccaa - 4b^3x - b^4 \\ \quad + 4bccx + 2bbcc \\ \qquad - c^4 \end{array}}{4bb - cc} = xx.$$

i. e. dividing the affected quantities by 4 both above and underneath,

$$\frac{\begin{array}{l} -b^3x \\ +bccx \end{array}}{bb - cc} + \frac{4ccaa - b^4 + 2bbcc - c^4}{4bb - 4cc} = xx$$

and actually dividing the former part by $bb - cc$

$$-bx + \frac{4ccaa - b^4 + 2bbcc - c^4}{4bb - 4cc} = xx.$$

Therefore according to the ſecond caſe,

$$-\frac{1}{2}b \dagger \sqrt{\frac{1}{4}bb \dagger \frac{4ccaa - b^4 \dagger 2bbcc - b^4}{4bb - 4cc}} = x;$$

or reducing $\frac{1}{4}bb$ to the ſame denomination with the reſt

$$-\frac{1}{2}b \dagger \sqrt{\frac{b^4 - bbcc \dagger 4ccaa - b^4 \dagger 2bbcc - c^4}{4bb - cc}} = x;$$

and leaving out thoſe quantities that deſtroy one another

$$-\frac{1}{2}b \dagger \sqrt{\frac{4ccaa \dagger bbcc - c^4}{4bb - cc}} = x.$$

The Arithmetical Rule. Multiply four times the ſquare of FB by the ſquare of the perpendicular CD, and add to it the product of the ſquare of EB into the □ FB, and from the ſum ſubſtract the biquadrate of FB; and the remainder will be the firſt thing found. Then ſubſtract four times the ſquare of FB from four times the ſquare of EB; and the remainder will be the ſecond thing found. Laſtly, divide the firſt thing found by the ſecond, and from the quotient take half after having extracted the root: Thus you'l have the leſſer ſegment of the baſe ſought, *&c. E. g.* In numbers you may put 2 for FB, 4 for EB, 12 for CD.

As for the *Geometrical Conſtruction*, the quantity of the laſt Equation contain'd under the radical ſign will help us to this proportion,

as $4bb - 4cc$ to $4aa \dagger bb - cc$ ſo cc to a fourth; or dividing all by 4,

as $bb - cc$ to $aa \dagger \frac{1}{4}bb - \frac{1}{4}cc$; ſo $\frac{1}{4}cc$ to a fourth, which is $\frac{1}{4}$ of the quantity under the radical ſign. Aſſuming therefore the quantity c for unity, make (*n.* 3.) IK = c IN and KL = b; NO will be = bb, and ſubſtracting OP = cc (*i. e.* to unity) there will remain NP = $bb - cc$. In like manner IS and KM = a, ST will = aa; to which if you add SX = $\frac{1}{4}$NO, and take thence XV = $\frac{1}{4}$ unity; TV will be = $aa \dagger \frac{1}{4}bb - \frac{1}{4}cc$. Wherefore if you make NR equal to this TV, and PQ = $\frac{1}{4}$ of unity or cc, ſince NP is = $bb - cc$; by the rule of proportion there will come out DR $\frac{1}{4}$ of that quantity, which is under the radical ſign. Therefore this being taken four times will give DZ for the whole quantity; to which if you join Dy = to unity, and, having deſcribed a ſemi-circle upon the whole line YZ, erect the perpendicular DE; this will be the root of the ſaid quantity, and taking hence moreover EF = $\frac{1}{2}b$, you'l have DE or DA the leſs ſegment of the baſe ſought. Therefore adding GB = b to DG, DB will be the greater ſegment, and, having let fall the perpendicular DC = a, BC and AC will be the ſides ſought. Q. E. F.

IV. *Some Examples of Affected Biquadratick Equations, but like Affected Quadratick ones.*

PROBLEM I.

To find a ſquare ABCD (ſuch as in the mean while we'll ſuppoſe n. 1. to be in Fig. 41.) from which having taken away another ſquare AEFG, which ſhall be half the former, there will be left the Rectangle GC whoſe Area is given. E. g. Suppoſe the given area equal to the ſquare of the given line LM, to find the true ſides of the ſquares AB and AE, anſwering to theſe ſuppoſed ones, *n.* 1.

SOLUTION.

Make the area of the rectangle that is to remain = bb, and GB = x; BC or AB will be = $\frac{bb}{x}$, and ſubſtracting hence GB, the remaining ſide of the leſſer ſquare AG = $\frac{bb}{x} - x$, *i. e.* $\frac{bb - xx}{x}$. Since therefore the ſquare of this is ſuppoſed to be half of the ſquare of AB, this will be the Equation:

$$\frac{b^4 - 2bbxx \dagger x^4}{xx} = \frac{b^4}{2xx};$$

and multiplying by xx,

$$b^4 - 2bbxx \dagger x^4 = \frac{b^4}{2}$$

and multiplying by 2,

$$2b^4 - 4bbxx \dagger 2x^4 = b^4;$$

and ſubſtracting $2b^4$, and adding $4bbxx$,

$$2x^4 = 4bbxx - b^4;$$

and dividing by 2,

$$x^4 = 2bbxx - \frac{b^4}{2}.$$

NB. The ſame Equation may be obtain'd, if, putting x for GB or FH, and having found the □ of AG or GF as above, you infer

$$\frac{b^4 - 2bbxx + x^4}{xx} = 2bb - xx.$$

This laſt Equation, tho' it be a biquadratick, yet may be rightly eſteem'd only a quadratick one, becauſe there is neither x^3 nor ſingle x in it, and ſo you may ſubſtitute this for it, $yy = 2bby - \frac{b^4}{2}$, *viz.* by ſuppoſing $y = xx$. Whence according to the third caſe of affected quadraticks,

y will $= bb + \sqrt{b^4 - \frac{b^4}{2}}$ i. e. $\sqrt{\frac{b^4}{2}}$

or $= bb - \sqrt{\frac{b^4}{2}}$

Therefore $x = \sqrt{bb + \sqrt{\frac{b^4}{2}}}$.

Geometrical Conſtruction. Now if the given line b be aſſumed for unity, bb and b^4 will be $=$ $=$ to the ſame line. Therefore, if between LM as unity, and MN $= \frac{1}{2}b$ viz $\frac{b^4}{2}$ you find a mean proportional MO (*n.* 2. *Fig.* 41.) that will be $= \sqrt{\frac{b^4}{2}}$, which being ſubſtracted from LM, and added to it, will give the two values of the quantity y. Moreover therefore by extracting its roots, *i. e.* by finding other mean proportionals LR and LS between the quantities found LP and LQ and unity (*n.* 3.) they will be the two values of the quantity x ſought; the firſt whereof LR will ſatisfie the queſtion, and the other LS be impoſſible. Wherefore to form the ſquare it ſelf, ſince its ſide will be $= \frac{bb}{x}$; by making (*n* 4.) as x to b ſo b to a fourth, it will be obtain'd: And this may be further prov'd, if finding a mean proportional BK between BI $=$ LR and the ſide of the □ BC, it be equal to the given quantity LM.

Arithmetical Rule. From the given area or the ſquare of the given line LM ſubſtract the root of half the biquadrate of the ſame line; thus you will have the value of the □ FC, *viz.* xx: Therefore extracting further the ſquare root of this, it will be the value of x ſought.

PROBLEM II.

TO *find another ſquare ABCD* (*Fig.* 42. *n.* 1.) *out of the middle whereof if you take another ſquare EFGH, which ſhall be a fourth part of the former, the area of the rectangle BK intercepted between BC and FG prolonged, ſhall be equal to the ſquare of a given line LM*; i. e. having theſe given to find the ſegment BI, and conſequently alſo the ſide BC or AB.

SOLUTION.

Make the area of the given rectangle, or the ſquare of LM $=$ to bb, and the ſide ſought of the rectangle BI $= x$; the other ſide BC will be $= \frac{bb}{x}$, and having ſubſtracted out of it IF and GK (*i. e.* $2x$) the ſide of the leſſer ſquare FG will $= \frac{bb}{x} - 2x$, i. e. $\frac{bb - 2xx}{x}$; whoſe ſquare ſince it is the fourth part of the greater ſquare by the Hypotheſis, you'l have

$$\frac{4b^4 - 16bbxx + 4x^4}{xx} = \frac{b^4}{xx};$$

and multiplying both ſides by xx,

$$4b^4 - 16bbxx + 4x^4 = b^4;$$

and taking away $4b^4$, and adding $16bbxx$,

$$4x^4 = 16bbxx - 3b^4;$$

and dividing by 4,

$$x^4 = 4bbxx - \tfrac{3}{4}b^4.$$

Therefore according to the third caſe of affected quadratick Equations,

$$xx = 2bb \pm \sqrt{4b^4 - \tfrac{3}{4}b^4} \text{ i. e.}$$

$$= 2bb \pm \sqrt{3\tfrac{1}{4}b^4}.$$

Therefore $x = \sqrt{2bb \pm \sqrt{3\tfrac{1}{4}b^4}}$.

Geometrical Conſtruction. If the given line b be taken for unity, b^4 and bb will be equal to it. Therefore if between LM as unity, and MN $= 3\frac{1}{4}b$, you find a mean proportional MO (*n.* 2. *Fig.* 42.) 'twill be $\sqrt{3\frac{1}{4}b^4}$; which ſubſtracted from MQ $= 2b$, or added to it, will give two values of the quantity xx, *viz.* PQ and IQ; the firſt whereof will be only a true one, and of uſe here. Now therefore a mean proportional QR found between PQ and unity will expreſs the quantity ſought x.

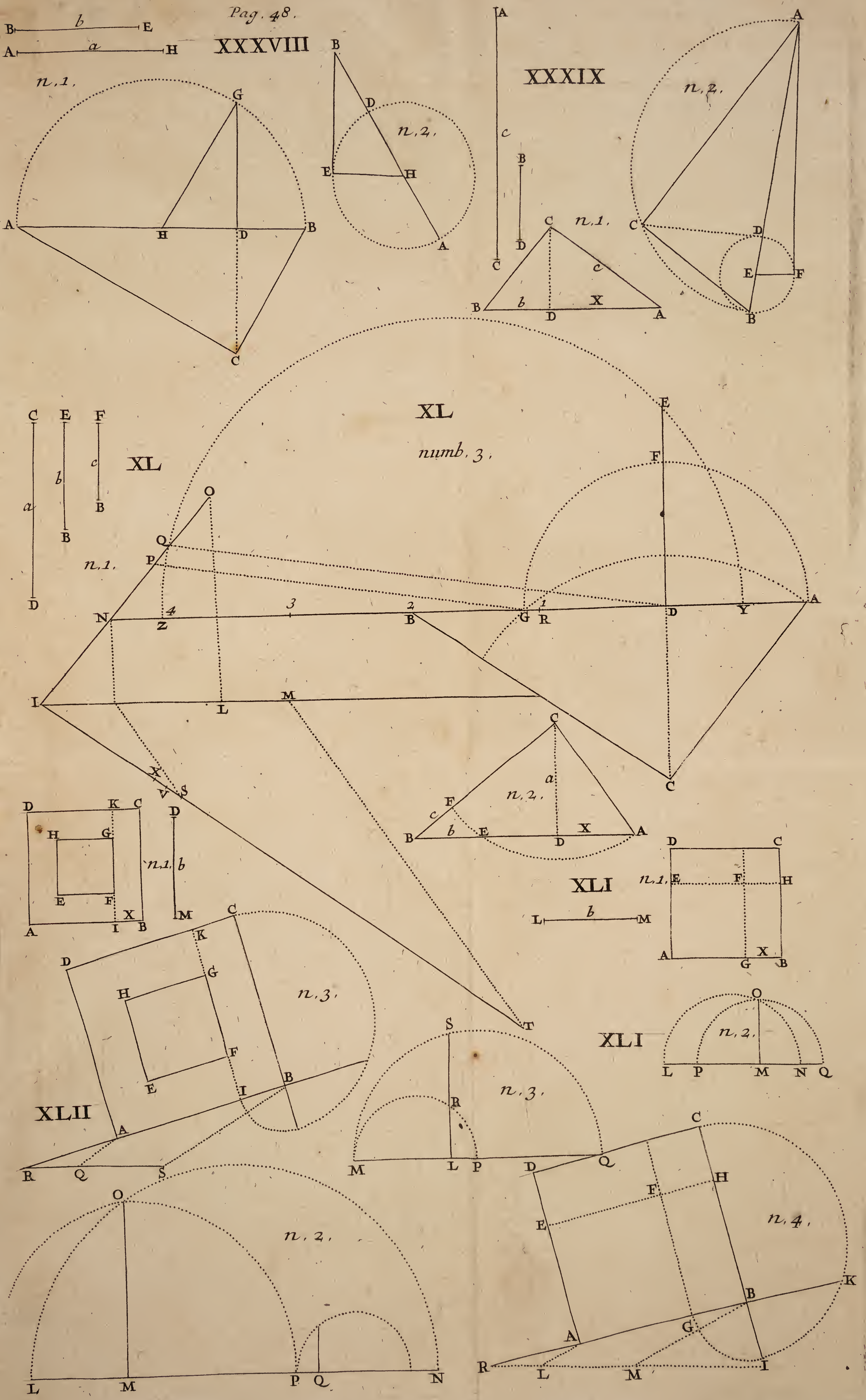
Therefore for forming the ſquare it ſelf, ſince its ſide AB is $= \frac{bb}{x}$, you may proceed as in the former Conſtructon, (*vid.* *n.* 3.)

PROBLEM III.

Having given the baſe of a right-angled Triangle, and a mean proportional between the Hypothenuſa and Perpendicular, to find the Triangle. As if the given baſe be AB (*Fig.* 43.) and the mean proportional between AC and BC be CD; to find the perpendicular BC, and Hypothenuſa AC.

SOLUTION.

Make the given baſe $= a$, and the mean proportional $= b$, the perpendicular BC $= x$, then will the Hypothenuſa be by the Hypoth, $\frac{bb}{x}$.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XXXIX
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XL
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n.2.
n.3.
XLII
n.1.
n.2.
n.3.
n.4.
A B C D E F G H I K L M N O P Q R S T V X Y Z
a b c
1 2 3 4

Therefore

$$\frac{b^4}{xx} - aa + xx;$$

and multiplying both ſides by xx,

$$b^4 = aaxx + x^4;$$

and ſubſtracting $aaxx$,

$$b^4 - aaxx = x^4.$$

Therefore by the ſecond caſe of affected quadraticks

$$xx = -\tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + b^4},$$

$$\text{and } x = \sqrt{-\tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + b^4}}.$$

Or thus.

Make the Hypothenuſa AC $= x$, then will the perpendicular be BC $= \frac{bb}{x}$. Therefore

$$xx = aa + \frac{b^4}{xx};$$

and multiplying by xx,

$$x^4 = aaxx + b^4.$$

Therefore by caſe 1.

$$xx = \tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + b^4}$$

$$\text{and } x = \sqrt{\tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + b^4}}.$$

Geometrical Conſtruction; the firſt for the latter Equation. If a be put for unity, the line AB will be alſo $= aa$, and making, as a to b ſo b to a third, *i. e.* as LM to MN ſo LO to OP, and you'l have bb. Having erected the perpendicular AQ $=$ OP upon AM, and drawn MQ or Mn equal to it; $\sqrt{\tfrac{1}{4}a^4 + b^4}$, and conſequently A$n$ will be $= \tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + b^4}$, *i. e.* the value of xx. Moreover a mean proportional AC found between An and AR unity will be the value of x, *i. e.* the Hypothenuſa ſought; which being found, you may eaſily complete the Triangle ABC.

2. In the caſe of the former Equation, making every thing as before AK would be the value of the quantity xx, *i. e.* $-\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^4 + b^4}$. Therefore a mean proportional RT found between RS $=$ AK and AR unity will be the value of x, *i. e.* the

i. e. the perpendicular ſought, and ſo AT the Hypothenuſa of the Triangle ſought.

Arithmetical Rule. In the firſt Solution add the biquadrate of the given mean proportional to the biquadrate of half the given baſe; and having extracted the ſquare root of the ſum, take from it half the ſquare of the given baſe; the root of the remainder will give the perpendicular of the triangle ſought, and the root of the ſum will give the hypothenuſa of it.

PROBLEM IV.

Having the Hypothenuſa of a right-angled Triangle given, and a mean proportional between the ſides to find the Triangle. As if the hypothenuſa be AC (*Fig.* 44.) and a mean proportional between the ſides BD, to find the ſides AB and BC.

SOLUTION.

Make the given Hypothenuſa $= a$, and the mean proportional $= b$, and the perpendicular BC $= x$; the baſis AB by the hypoth. will be $\frac{bb}{x}$. Therefore

$$\frac{b^4}{xx} + xx = a;$$

and multiplying by xx,

$$b^4 + x^4 = aaxx;$$

and ſubſtracting b^4,

$$x^4 = aaxx - b^4;$$

Therefore by the third caſe,

$$xx = \tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 - b^4}$$

$$\text{and } x = \sqrt{\tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 - b^4}}.$$

Geometrical Conſtruction. If a be put for unity, AC will be alſo $= aa$, and by making as AC to CG (a to b) ſo AF to GH (b to a third) this third will be GH $= bb$. Aſſuming therefore OC $= \frac{1}{2}aa =$ OB the radius of a ſemi-circle, and having erected CD $= bb =$ BE parallel to it, EO will be

$\sqrt{\tfrac{1}{4}a^4 -}$

$\sqrt{\frac{1}{4}a^4 - b^4}$, and conſquently EC $= \frac{1}{2}a - \sqrt{\frac{1}{4}a^4 - b^4}$, and EA $\frac{1}{2}a + \sqrt{\frac{1}{4}a^4 - b^4}$, *viz.* the double value of the quantity xx. Therefore for the double value of x, you muſt extract the roots out of them, *i. e.* you muſt find the mean proportionals AL and AM between unity AC and AI = EC on the one ſide, and AK = AE on the other; Altho' theſe laſt may be more compendiouſly had, and the triangle it ſelf immediately conſtructed, if having found EC and EA, you draw CB and AB: For theſe will be thoſe two laſt mean proportionals = = AL and AM; for by reaſon of the △ △ ABC, AEB, and BEC, BC is a mean proportional between AC and CE, and AB a mean proportional between the ſame AC and AE by the 8. *Lib.* 6. *Eucl.* which is *Conſect.* 3. *Schol.* 2. *Prop.* 34. *Lib.* 1. *Math. Enucl.*

PROBLEM V.

HAving given the Area and Diagonal of a right-angled Parallelogram, to find the ſides and ſo the Parallelogram. As if the given Area be = to the ſquare of a given line BD (*Fig.* 45) and the Diagonal AC, to find the ſides AB and BC.

SOLUTION.

If for the given Area, or ſquare of the line BD you put bb, and make the Diagonal AC $= a$, and put for the leſſer ſide BC, x; the other ſide will be $\frac{bb}{x}$.

Therefore $\frac{b^4}{xx} + xx = aa$;

and multiplying by xx,

$b^4 + x^4 = aaxx$; and ſubſtracting b^4,

$x^4 = aaxx - b^4$. Which Equation, ſince it is the ſame with that of the preceding Problem (which is no wonder, ſince this fifth perfectly coincides with the fourth; for the Diagonal AC is the hypothenuſa, and BD, whoſe ſquare is = to the given area of the rectangle, is a mean proportional between the ſides AB and BC) and ſo will have the ſame Conſtruction,

ſtruction, (*ſee Fig.* 45.) and the ſame Arithmetical Rule, which may be eaſily formed from the laſt Equation of the preceding Problem.

PROBLEM VI.

H*Aving given the firſt of three proportional lines, and another whoſe ſquare is equal to both the ſquares of the other two, to find thoſe two proportionals.* As if AC (*Fig.* 46.) be the firſt of the three proportionals, and another line ED given, whoſe ſquare equals the two ſquares of the others taken together; to find thoſe two as ſecond and third proportionals.

SOLUTION.

If for AC you put a, and make the given line ED $= c$, and the ſecond proportional $= x$, the third will be $\frac{xx}{a}$. Wherefore the ſquares of the two laſt will be $\frac{x^4}{aa} + xx = cc$, □ ED by the hypoth. and multiplying both ſides by aa.

$x^4 + aaxx = aacc$;

and ſubſtracting $aaxx$,

$x^4 = -aaxx + aacc$. Therefore

$xx = -\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + aacc}$

and $x = \sqrt{-\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + aacc}}$.

Geometrical Conſtruction. If a be put for unity, AC will alſo $= aa$ and a^4, and making as AC to CD (a to c) ſo AF to DE (c to a third) DE will be $= cc$. Now having made AK $=$ DE, *i. e.* cc, a mean proportional AI found between AC and AK will be $\sqrt{aacc}$. Therefore taking AO $= \frac{1}{2}$AC, *viz.* $\frac{1}{2}aa$, the hypothenuſa OI will be $= \sqrt{\frac{1}{4}a^4 + aacc}$. And OA $= \frac{1}{2}a$ being ſubſtracted from OI or OH equal to it will leave AH the value of xx; and the root of that being extracted, *i. e.* finding another mean proportional AG between AC and AH, it will be the value of x, *i. e.* the ſecond of the proportionals ſought, and ſince AC is the firſt given, AH will be the third. Q. E. F.

NB. This Conſtruction may be abbreviated, and the firſt Operation, by which you find DE, to which afterwards AK is made equal, may be omitted. For ſince you make uſe of a mean proportional between CA and DE ſought, which is $=$ to the given line ED, and afterwards AI a mean proportional between AC and AK is ſought; it is evident that AI will be equal to ED given, and conſequently that they may be immediately joined at right angles at the beginning of the given line AC, and the reſt may then be done as before.

Some Examples of Cubick and Biquadratick Equations, both ſimple and affected, whether reducible or not.

PROBLEM I.

Between two given right angles to find two mean proportionals. E. g. Suppoſe given AB the firſt and CD the fourth, (*Fig.* 47. *n.* 1.) between theſe to find two mean proportionals.

SOLUTION.

Make the firſt of the given quantities AB $= a$, the other CD $= q$, the firſt mean $= x$, then will the latter be $\frac{xx}{a}$, and conſequently $\frac{x^3}{aa} = q$; and multiplying by aa, $x^3 = {\ast\ast}\, aaq$.

The Central Rule will b $\frac{L}{2} =$ AD

$\frac{r}{2L^2} =$ DH. i e. according to a ſuppoſition we ſhall by and by make, $\frac{1}{2}a =$ AD, and $\frac{1}{2}q =$ DH.

Geometrical Conſtruction. If AB or a be made unity, and alſo the *Latus Rectum* of your Parabola, and you deſcribe, by means of this *Latus Rectum*, the Parabola, according to *Schol.* 1. *Prop.* 1. *Lib.* 2. *Math. Enucl.* [ſee *n.* 2. and 3. *Fig* 47.] in which AB is the *Latus Rectum*; A1, A2, *&c.* the Abſciſſa's; AI, AH, *&c.* the ſemiordinates; make moreover (*n.* 4) AD $= \frac{1}{2}a$, and having from D erected a perpendicular $= \frac{1}{2}q$, deſcribe a circle at the interval AH, cutting the parabola in N: Which being done, a perpendicular to the Ax will be the root ſought or the value of x, *i. e.* the firſt of the means, and conſequently OA the other; ſince NO by the firſt property of the Parabola (ſee *Prop.* 1. *Lib.* 2. *Matheſ. Enucl*) is a mean proportional between the *Latus Rectum* AB, and the abſciſſa AO. And by this means there will come out, by *Baker*'s Central Rule the very conſtruction of *Des Cartes*, *Geom.* p. m. 91.

The Arithmetical Rule. Multiply the ſquare of the firſt by the fourth given, and the cube root extracted out of the product, will expreſs the firſt of the means ſought.

PROBLEM II.

H*Aving given the ſolid Contents of a ſolid or an hollow Parallelepiped, and the proportion of the ſides, to find the ſides.* As, if the given capacity or ſolid contents be = to the cube of a certain given line IK (*Fig.* 48. *n.* 1.) and the proportion of the heighth to the length be as AB to BC, and to the latitude as the ſame AB to BD; to find firſt the altitude, which being had, the other Dimenſions will alſo be known, by the given proportions.

SOLUTION.

Make IK $= a$, AB $= b$, BC $= c$, and BD $= d$; and laſtly the heighth ſought $= x$, then

as b to c, ſo x to the length required $\frac{cx}{b}$;

and as b to d, ſo x to the latitude ſought $\frac{dx}{b}$.

Multiplying therefore theſe three dimenſions of the Parallelepiped together, you'l have its capacity or ſolid contents $\frac{cdx^3}{bb}$ $= a^3$;

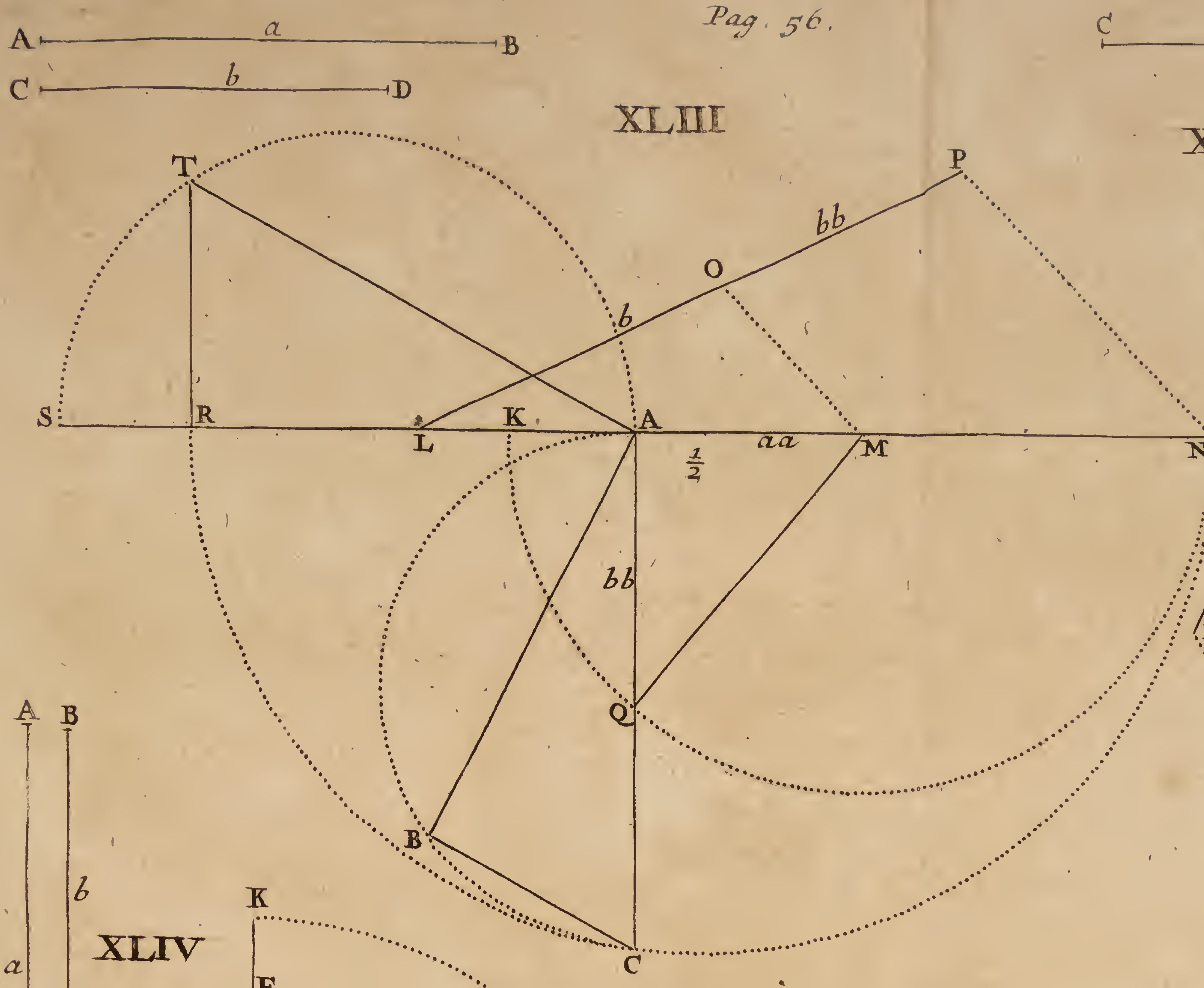
and multiplying by bb,

$cdx^3 = a^3bb$; and dividing by cd,

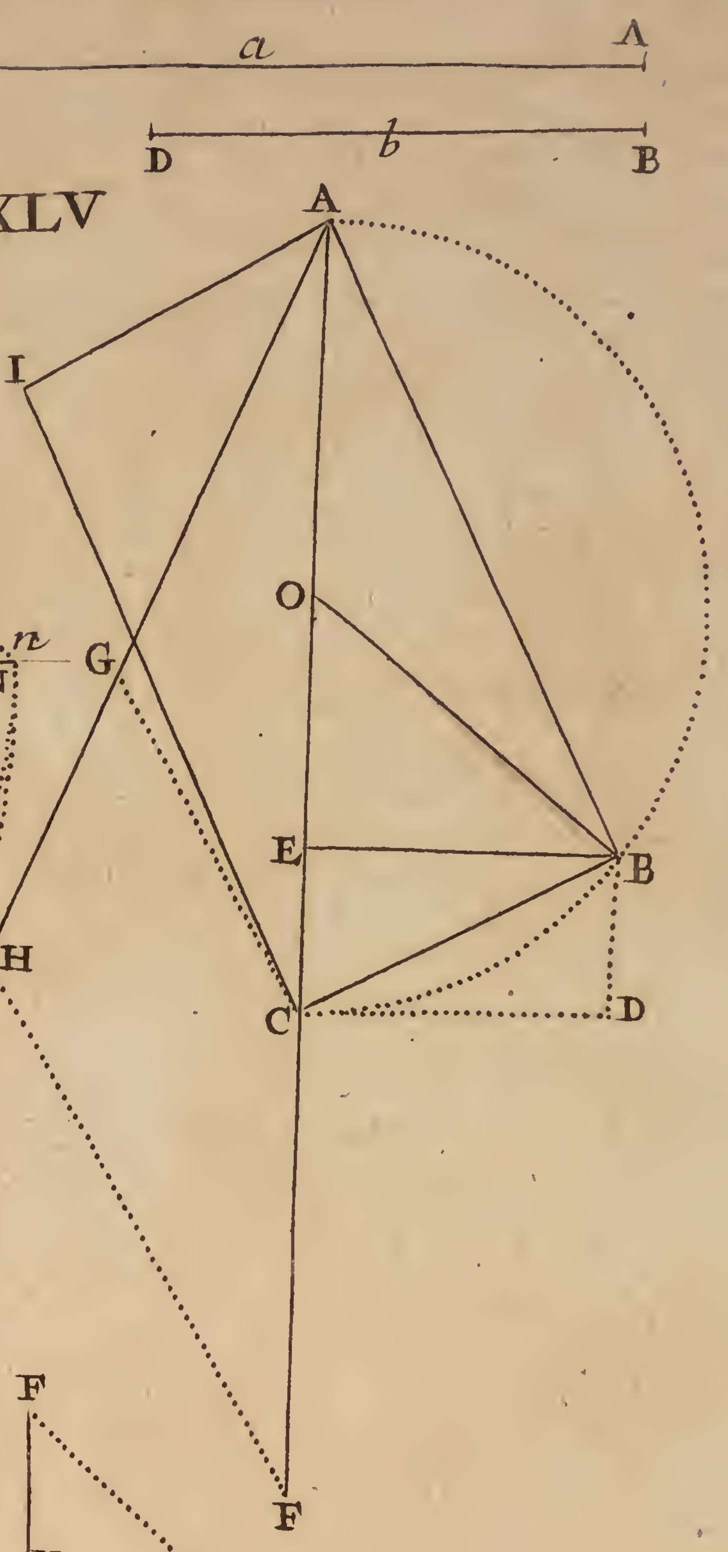
$x^3 = \frac{a^3bb}{cd}$ i e. $x^3 \ast\ast - \frac{a^3bb}{cd} = 0$.

There-

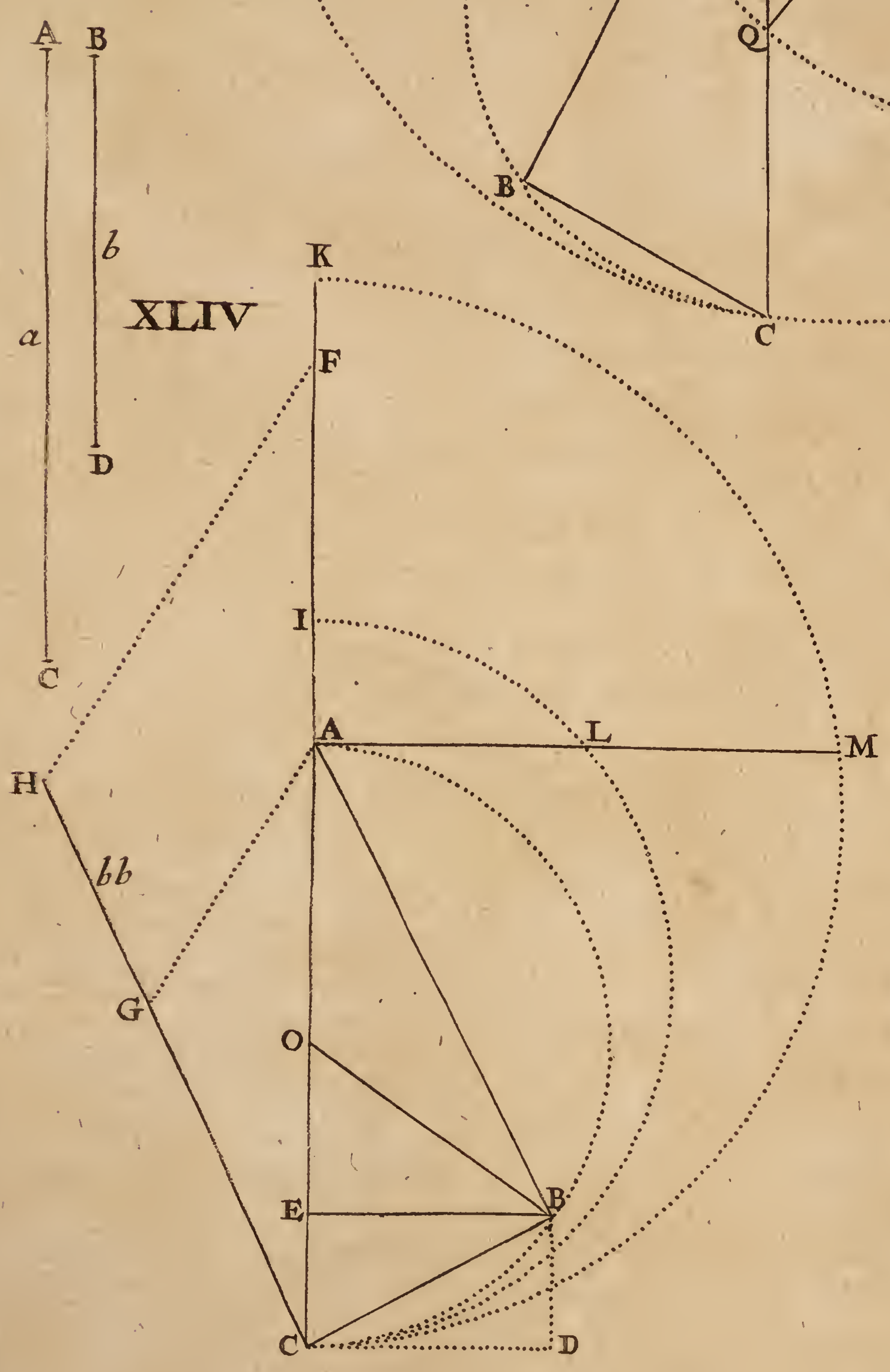
XLIII



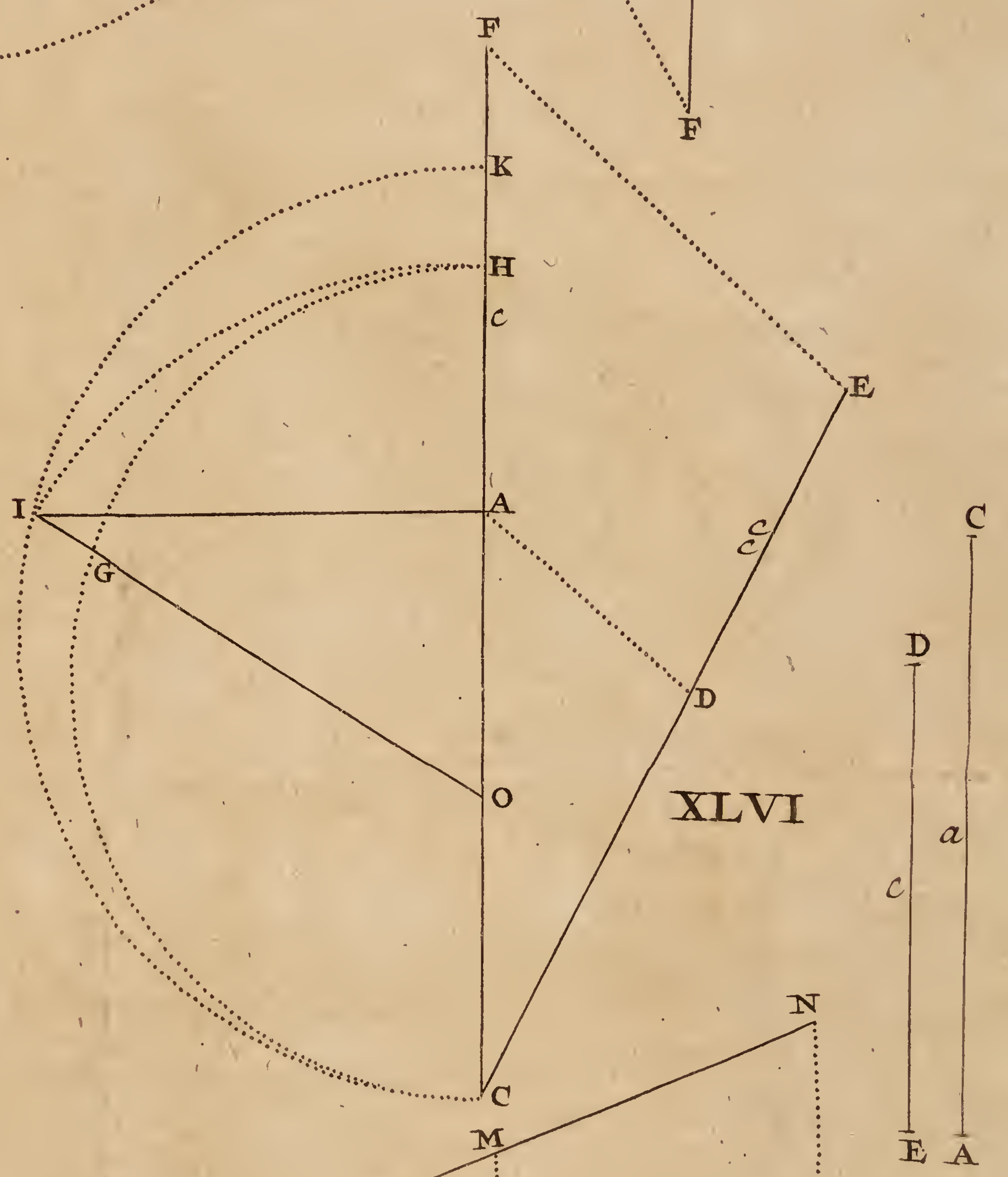
XLV



XLIV



XLVI



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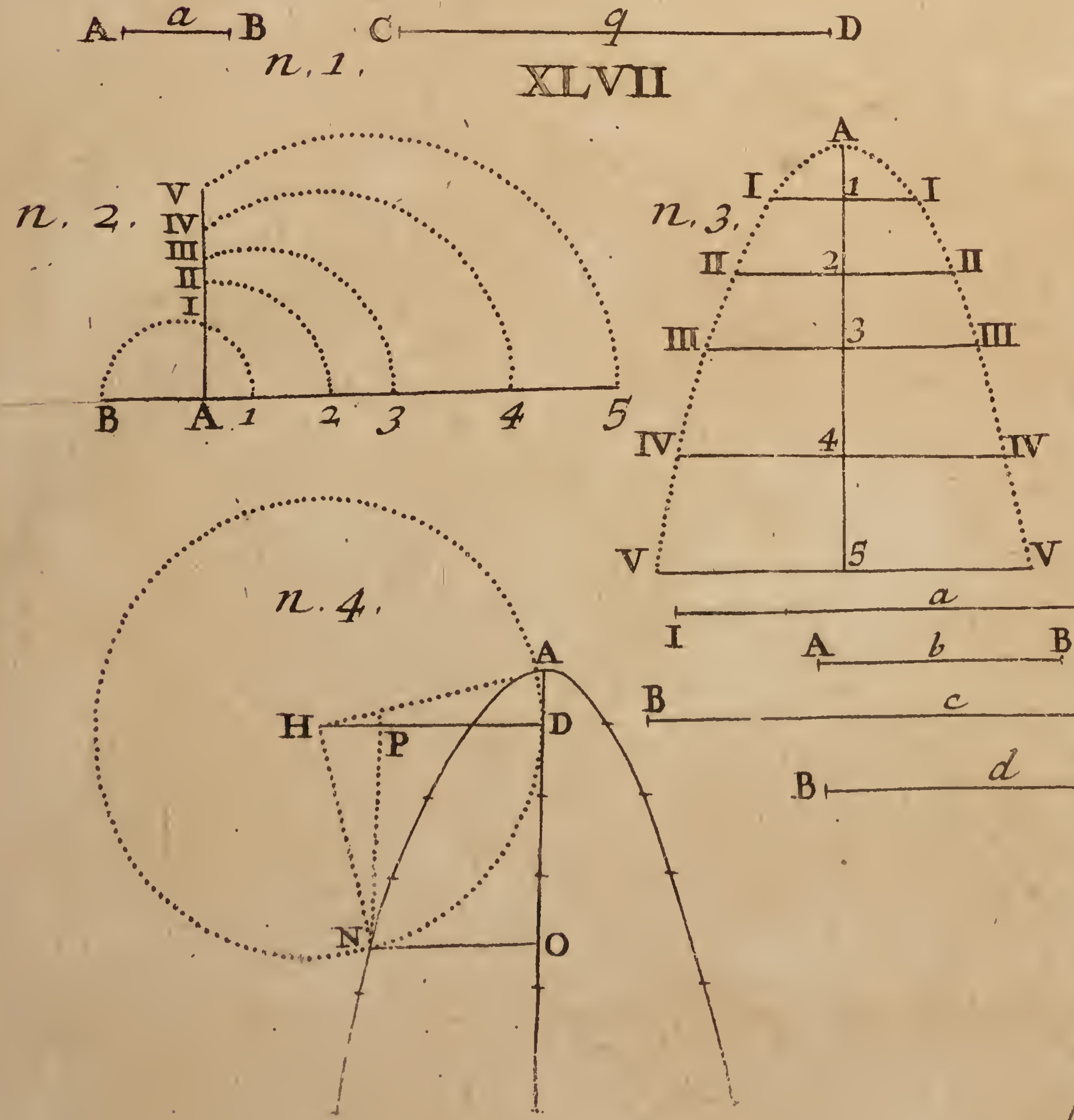
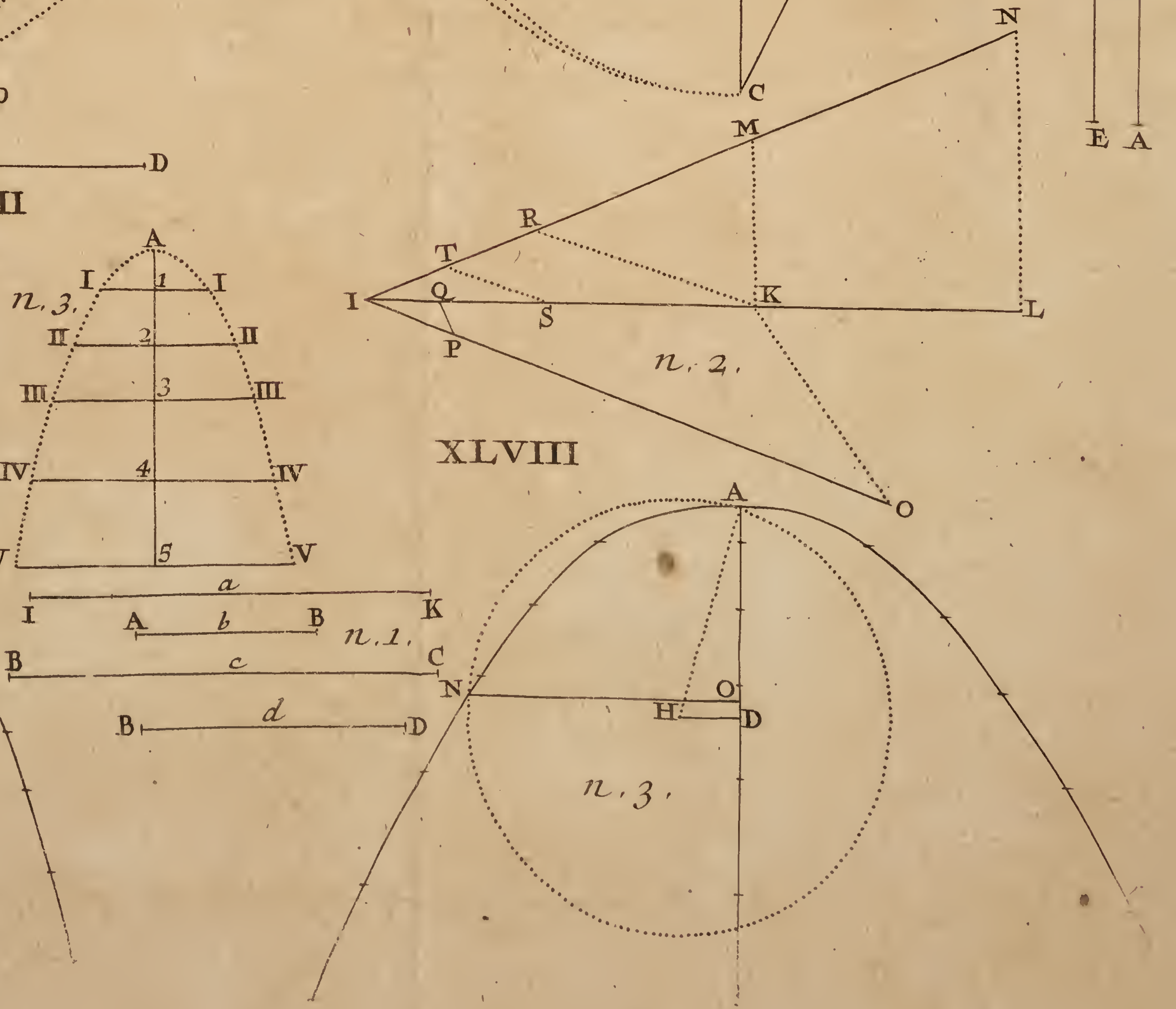


XLVIII



Therefore the Central Rule will be the ſame as above,

$\frac{L}{2} = AD$ and $\frac{r}{2L^2} = DH$, *i. e.* according to the ſuppoſition which will by and by follow,

$\frac{a}{2} = AD$ and $\frac{a^3bb}{2cd} = DH$.

Geometrical Conſtruction. If IK or *a* be made unity, and at the ſame time *Latus Rectum*, and by means of it you deſcribe a Parabola, after the way we have ſhewn, *Fig.* 47. *n.* 2. and 3. and ſhall always hereafter make uſe of; and then to prepare the quantity $\frac{a^3bb}{2cd}$ (which in the Central Rule is the the quantity $\frac{r}{2}$) make (*n.* 2.)

IK—IM = BC—KL = BD—MN
as *a* to *e* ſo *d* to *e*;

ſo that for *cd* you may put *ae*, and afterwards divide by *a* both above and underneath; you'l have the quantity $\frac{r}{2} = \frac{aabb}{2e}$.

Therefore by further inferring
as 2*e* to *bb*, ſo *aa* to a fourth
IO—IP = IT—IK—IQ, and you'l have the quantity DH, which will determine the centre, after AD is made equal $\frac{a}{2}$. Having therefore deſcribed from that centre a circle through the vertex of the Parabola A (*n* 3.) a ſemi-ordinate NO drawn from the interſection will be the altitude ſought, which will eaſily give you the length and breadth by the reaſons above ſhewn.

Another Solution.

Which will be more accommodated to the Arithmetical Rule.

Let the reſt of our Poſitions or *Data* remain as above, but the name of the proportion which the altitude has to the length be = *e*, and of that which it has to the latitude = *i*, the length will be $=$ to ex, and the latitude to ix. Wherefore multiplying the ſides together, you'l have the whole ſolidity,

$eix^3 = a^3$; and dividing by ei,

$x^3 = \frac{a^3}{ei}$. Therefore

$x = \surd C.\, \overline{\frac{a^3}{ei}}$. Hence

The Arithmetical Rule. Multiply together the given names of the reaſons, and divide the given cube by the product; which done, the cubick root extracted out of the quotient will be the altitude of the ſolid ſought.

Another *Geometrical Conſtruction.* Now if we would alſo conſtruct this Equation $x^3 = \frac{a^3}{ei}$ geometrically, putting AB $= b$ for unity, BC and BD will be the names of the reaſons $= e$ and i. Making therefore firſt

IK——IM——KL ——MN

as a to e ſo i to a fourth f (*Fig.* 49. *n* 1.) af will be $= ei$, and the propoſed Equation will have this form:

$x^3 = \frac{a^3}{af}$ i. e. $\frac{aa}{f}$

Therefore 2. making

as f to a ſo a to a third IQ, that will be the value of x^3. But hence 3. by extracting the cube root, *i. e.* by finding two mean proportionals between unity b, *viz.* AB and the line found IQ; the firſt of them will give the root ſought.

NB. The ſame Central Rule would come out according to *Baker*'s Central Rule, which would have the ſame form of the Equation, as the precedent Example $x^3 ** - aa = 0$, i. e.

taking b for the *Lat. Rect.* and unity, $\left\{ \begin{array}{l} \frac{L}{2} = AD \text{ and } \frac{r}{2L^2} = DH, \\ \frac{b}{2} = AD \text{ and } \frac{aa}{2f} \text{ i. e. } \frac{1}{2}\, IQ = DH. \end{array} \right.$

(ſee *Fig.* 49. *n.* 2.)

PRO-

PROBLEM III.

Having given the ſolid contents of a ſolid or hollow Parallelepiped, and the difference of the ſides, to find the ſides. As, if the given capacity be equal to the cube of any given line LM (*Fig.* 50. *n.* 1.) and the difference whereby the length exceeds the breadth ≍ NO, and the difference by which the breadth exceeds the altitude or depth ≍ PQ, to find the length, breadth and depth.

SOLUTION.

Make the ſide of the given cube ≍ a, the exceſs of the length above the breadth NO ≍ b, and of the breadth above the depth PQ ≍ c, make the depth ≍ x, the breadth ≍ $x \dagger c$, and the length ≍ $x \dagger b \dagger c$. Multiplying therefore theſe three dimenſions together.

$$\begin{array}{rl}
\text{Length} & x \dagger b \dagger c \\
\text{Breadth} & x \dagger c \\
\hline
 & xx \dagger bx \dagger cx \\
 & \qquad cx \dagger cb \dagger cc \\
\hline
 & xx \dagger 2cx \dagger bx \dagger cb \dagger cc \\
\text{Depth} & x \\
\hline
 & x^3 \begin{matrix} \dagger 2c \\ \dagger b \end{matrix}\Big\} x \begin{matrix} \dagger cb \\ \dagger cc \end{matrix}\Big\} x \asymp a^3
\end{array}$$

Or according to the forms of *Baker* and *Cartes*

$$x^3 \begin{matrix} \dagger b \\ \dagger 2c \end{matrix} xx \begin{matrix} \dagger bc \\ \dagger cc \end{matrix} x - a^3 \asymp o.$$

Wherefore the Central Rule contracted by the ſuppoſition which will hereafter follow will be this,

$$\frac{L}{2} \dagger \frac{p^2}{8} \backsim \frac{q}{2} \asymp \text{AD}$$

and $\frac{p}{4} \dagger \frac{p^3}{16} \backsim \frac{pq}{4} \backsim r \asymp \text{DH}.$

i. e. by virtue of the ſuppoſition juſt now mentioned (which takes LM *viz.* a for unity and alſo for *Lat. Rectum*)

a †

$\frac{a}{2} \dagger \frac{bb \dagger 4bc \dagger 4cc}{8} \backsim \frac{bc \backsim cc}{2} = AD.$ And

$\frac{b \dagger 2c}{4} \dagger \frac{b^3 \dagger 6bbc \dagger 12bcc \dagger 8c}{16} \backsim \frac{bbc \backsim 3bcc \backsim 2c^3}{4}$

$(\backsim \frac{a^3}{2} = DH.$

Or more ſhort;

$\frac{a}{2} \dagger \frac{bb}{8} \dagger \frac{bc \dagger cc}{2} - \frac{bc - cc}{2}$ i. e. $\frac{a}{2} \dagger \frac{bb}{8} = AD$; and

$\frac{b \dagger 2c}{4} \dagger \frac{b^3}{16} \dagger \frac{1\frac{1}{2}bbc \dagger 3bcc \dagger 2c^3}{4} - \frac{bbc - 3bcc - 2c^3}{4}$

$(\backsim \frac{a^3}{2}$ i. e.

$\frac{b \dagger 2c}{4} \dagger \frac{b^3}{16} \dagger \frac{bbc}{8} \backsim \frac{a^3}{2} = DH.$

Geometrical Conſtruction. If LM or *a* be taken for unity or *Latus Rectum* of the Parabola to be deſcribed, that being deſcribed (*Fig.* 50. *n.* 3.) you are firſt of all to determine two quantities AD and DH; which may be done two ways; either by *Baker*'s form of his Central Rule, or by ours immediately divided by the quantities of the laſt Equation.

1. For AD by our form, $\frac{a}{2} \dagger \frac{bb}{8} = AD$, you muſt make (*Fig.* 50. *n.* 1.) as *a* to *b* ſo *b* to a third (AB to AC ſo BE to CF) which will be *bb*, and D^2 the eighth part of this CF muſt be added to A^2 the half of AB. And by *Baker*'s form you muſt make, 1. as AB ($= a$, *n.* 2.) to AC ($= \frac{1}{2}p$ i. e. $\frac{1}{2}b \dagger c$) ſo BE ($= \frac{1}{4}p$ i. e. $\frac{1}{4}b \dagger \frac{1}{2}c$) to a fourth CF (which will be $= \frac{p^2}{8}$:) 2. Make moreover as AB to AG (*a* to *b*) ſo BH to GI (*c* to *bc*) and, as AB to AK (*a* to *c*) ſo BH to KL (*c* to *cc*) and the two quantities found GI and KL (*bc* and *cc*) being added into one ſum will give the quantity $q =$ MN, the half whereof MO will expreſs the quantity $\frac{q}{2}$ in the

Rule, and to be ſubſtracted from the former $\frac{a}{2} \dagger \frac{p^2}{8}$. Actually therefore to determine the quantity AD not on the Ax, but on another Diameter of the deſcribed parabola *n.* 3. (becauſe the quantity p is in the Equation) having made a perpendicular to the Ax aE $= \frac{p}{4}$ *i. e.* to the line BE *n.* 2. and from E having drawn EA parallel to the ax, according to our form AD *n.* 1. transfer it only on the diameter of the parabola *n.* 3. from A to D, either by parts $\frac{a}{2}$ i. e. $\frac{1}{2}$LM from A to c, and $\frac{bb}{8}$ i. e. $\frac{1}{8}$CF *n.* 1. from c to D: But according to *Baker's* form, firſt you muſt put $\frac{a}{2} = \frac{1}{2}$ LM *n.* 1. from A to 1. Secondly you muſt put from 1 to 2 the quantity $\frac{p^2}{8} =$ CF *n.* 2. Thirdly you muſt put from 2 to 3 backwards the quantity to be ſubſtracted $\frac{q}{2} =$ MO *n.* 2. which being done, the point D will be determined.

[*It is evident by comparing theſe two ways of Conſtruction, that we may join our forms not incommodiouſly to* Baker's; *becauſe by ours the quantity AD was obtained more compendiouſly than by* Baker's, *which will alſo often happen hereafter. And where this Compendium cannot be had, there is another not inconſiderable one, that, if the quantities AD and DH determined according to both ways ſhall coincide, (which happens in the preſent caſe) we may be ſo much the more ſure of the truth*]

2. As for DH by our form, you muſt put it from D to e on a perpendicular erected from D on the left hand, the quantity $\frac{b \dagger 2c}{4} =$ BE *n.* 2. falling here upon the Ax. Then for the quantity $\frac{b^3}{16}$ make *n.* 4. as a to $\frac{1}{2}bb$ (LM to LN) ſo $\frac{1}{8}b$ to $\frac{b^3}{16}$ (MO to NP) and this quantity muſt be put from e to f in a perpendicular to the Diam. Thirdly, for the quantity $\frac{bbc}{8}$

yo

you muſt farther make *n.* 4. as a to $\frac{1}{2}bb$ (DM to LN) ſo $\frac{1}{4}c$ to a fourth (MQ to NR) which muſt be put from f to g. Laſtly the quantity $\frac{a^3}{2}$ (which is $=$ MO *n.* 2.) muſt be put backwards (becauſe to be ſubſtracted) from g to H, which is the centre ſought. In like manner by *Baker*'s form, firſt $\frac{p}{4} =$ BE is put from D to 1 even to the Ax. Secondly, for the quantity $\frac{p^3}{16}$ make, *n.* 4. as a to $\frac{p^2}{8}$ (LM to LS $=$ CF *n.* 2.) ſo $\frac{p}{2}$ to a fourth (MT $=$ AC *n.* 2. to SV) and this SV is further put (*n:* 3.) from 1 to 2. Thirdly, make in the ſame *Fig. n.* 4. for the quantity $\frac{pq}{4}$, as a to $\frac{p}{2}$ (LM to MT) ſo $\frac{q}{2}$ to a fourth (LX to XZ) and this XZ is put backwads (*n.* 3.) from 2 to 3, which preciſely coincides with the point g. Laſtly the remaining quantity $\frac{r}{2}$ ($=$ MO *n.* 2. and ſo by what we have ſaid above, preciſely coinciding with the interval gH) is put backwards from g to H the Centre ſought.

[*Hence it appears again that* Baker'*s form is more laborious than ours; tho' both accurately agree, and hereafter, for the moſt part, we ſhall uſe them both together, tho in the work it ſelf, rather in Figures, than in that tedious proceſs of words, which we have here for once made uſe of, that it might be as an Example for the following Conſtructions.*]

Having therefore found by one or both ways the Center H, and thence deſcribed a circle thro' the vertex of the parabola A, the interſection N will give the perpendicular to the diameter NO, the value of the quantity x ſought.

PROBLEM IV.

TO *divide a given angle NOP, or a given arch NQTP (Fig. 51. n. 1.) into three equal parts*; *i. e.* having given or aſſumed at pleaſure the Radius NO, and conſequently alſo the Chord of the arch NP, to find NQ the ſubtenſe of the third part of the given arch.

SOLUTION.

If NO be made unity, NP $= q$, and NQ be ſuppoſed $= z$; having drawn QS parallel to TO, you'l have three ſimilar triangles NOQ, QNR and RQS. For ſince the angle QOP is double of the angle QON, and the ſame (as being at the Center) double alſo of the angle at the Periphery QNR; it will be equal to the angle QON. But the angle at Q is common to both triangles: Therefore the whole are equi-angular, and conſequently the legs NQ and NR equal, as alſo NO and QO; and by the like reaſon alſo PY and PT. Wherefore if RS ſhould be added to RY, the line NP by this addition would be triple of the line NQ; and ſo would give the Equation, if RS was determined; which may be done by means of the △ QRS, ſimilar to the two former NOQ and QNR; for the angle RQS is equal to the alternate one QOT = QNR, and the angle at R common to the triangles QNR and RQS, *&c.* Wherefore

as NO to NQ ſo NQ to QR

1 —— z —— z —— zz

and as NQ to QR ſo QR to RS

z —— zz —— zz —— $\frac{z^4}{z}$

Therefore according to what we have above ſaid

$q + z^3 = 3z$; and ſubſtracting q

$z^3 = 3z - q$; or

$z^3 - 3z + q = 0$

Therefore the Central Rule will be (ſuppoſing alſo unity NO for the *Latus Rectum*,)

$\frac{L}{2} † \frac{q}{2} = AD$ *i. e.* by our form, $\frac{1}{2} † \frac{3}{2}$ *i. e.* $2NO = AD$

and $\frac{r}{2} = DH.$ and $\frac{q}{2} = DH.$

Geometrical Conſtruction. Having deſcribed your parabola (*Fig.* 51. *n.*2.) take on its Ax (becauſe the quantity *p* is wanting in the Equation) AD = 2NO, and from D having erected a perpendicular = $\frac{3}{2}$NP to H; that will be the Center, from which a circle deſcribed thro' A, by cutting the parabola in three places, will give the three roots of the Equation, *viz.* NO and *n o* true ones, the firſt whereof will expreſs the quantity ſought NQ (*n.* 1.) the latter the line NV, being the ſubtenſe of the third part of the compl. of the arch; and MO will expreſs the falſe root, which is equal to the former taken together: All the ſame as in *Cartes* p. 91. but here ſomewhat plainer and eaſier.

PROBLEM V.

Having three ſides given of a quadilateral Figure to be inſcribed in a circle, AB, BC, CD, (*Fig.* 52. *n.* 1.) *to find the fourth ſide, which ſhall be the Diameter of the Circle.*

SOLUTION.

If we conſider the buſineſs as already done, and make AB = *a*, BC = *b*, CD = *c*, and AD = *y*; we ſhall have firſt in the right-angled △ ABD, □ BD = *yy*—*aa*, and (ſince in the obtuſe angled △ BCD the □ BD is = □ □ BC † CD † 2 ▭ of BC into CE) if thoſe two □ □ BC † CD (*i. e. bb* † *cc*) be ſubſtracted from □ BD (*yy*—*aa*) you'l have 2. *yy*—*aa*—*bb*—*cc* = to the two ſaid rectangles of BC into CE. But theſe two rectangles may alſo be otherwiſe obtain'd, 3, if the ſegment CE be otherwiſe determined; which may be done by help of the ſimilar △ △ ABD and CED (for the angles at B and E are right ones, and ECD and BAD equal, becauſe each with the ſame third BCD makes two right ones; the one ECD by reaſon of its contiguity, the other at A by the 22. *Eucl. Lib.* 3.) *viz.* by inferring

as DA to AB ſo DC to CE

$$y \text{ — } a \text{ — } c \text{ — } \frac{ac}{y}$$

for now multiplying CE $= \frac{ac}{y}$ by BC $= b$, the ▭ of BC into CE will be $= \frac{abc}{y}$ and two ſuch $\frac{2abc}{y}$.

Now therefore $yy - aa - bb - cc = \frac{2abc}{y}$; and multipl. by y,

$$\left.\begin{matrix} y^3 * -aa \\ -bb \\ -cc \end{matrix}\right\} y = 2abc;$$ i. e. by *Baker*'s and *Cartes*'s forms.

$$\left.\begin{matrix} y^3 * -aa \\ -bb \\ -cc \end{matrix}\right\} y - 2abc = o.$$

Therefore the Central Rule will be (ſuppoſing the ſame quantity *e. g.* a for unity and *Lat. Rect.*)

$\frac{L}{2} \dagger \frac{q}{2} =$ AD and

$\frac{r}{2} =$ DH, *i. e.* according to our form,

$\frac{a}{2} \dagger \frac{aa \dagger bb \dagger cc}{2}$ i. e. $a \dagger \frac{bb \dagger cc}{2} =$ AD, and

Geometrical Conſtruction, which, without any circumlocution, from our form is founded on the foll. in *Fig.* 52. *n.* 2.

LM $= a$ *n.* 3. A 1. $=$ LM from *n.* 2.
MN and LP $= b$ 1, 2 $= \frac{1}{2}$ PQ
PQ $= bb$ 2, 2 $= \frac{1}{2}$ ST
LS and MO $= c$ DH $=$ PR
PR $= bc$ MO and *mo* two falſe roots
ST $= cc$ NO the true root; upon which having deſcribed a ſemi-circle the quadrilateral will be eaſily made. According to *Baker*'s form, for AD there would be *n.* 3. firſt

$Ac = \frac{1}{2}LM$, then $cD = \frac{q}{2} = VX$, half the line VZ, which is compounded of LM, PQ and ST; but DH is = PR as above.

PROBLEM VI.

Having given to form a right-angled Triangle the leaſt ſide BA (Fig. 53. n. 1.) and the difference of the ſegments of the baſe, to find the difference of the ſides, and ſo form the Triangle. If we repreſent the buſineſs as already done, having given AB and EC to find FC.

SOLUTION.

Make $AB = a$, and $EC = b$, and $FC = x$; then will $BC = a + x$: Therefore the □ $AC = 2aa + 2ax + xx$ and the line AC $\sqrt{2aa + 2ax + xx}$ and $\sqrt{2aa + 2ax + xx} - b$. Now therefore ACE *i. e.* $\sqrt{2aa + 2ax + xx}$ multiplyed by b or $\sqrt{bb}$ i. e. $\sqrt{2aabb + 2abbx + bbxx}$ is = GCF [but GC is $= 2a + x$] i. e. $2ax + xx$ by *Conſ.* 1. *Prop.* 47. *Lib.* 1. *Matheſ. Enucl.* and ſquaring both ſides

$$2aabb + 2abbx + bbxx = 4aaxx + 4ax^3 + x^4;$$

and tranſpoſing all,

$$x^4 + \underset{(p)}{4ax^3} + \underset{(q)}{\begin{array}{c}4aaxx\\ -bb\end{array}} \underset{(r)}{- 2abbx} \underset{(s)}{- 2aabb} = 0.$$

Therefore (taking a for 1 and the *Latus Rectum* of the parabola) the Central Rule will be,

$$\frac{L}{2} + \frac{p^2}{8} - \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} - \frac{pq}{4} - \frac{r}{2} = DH$ *i. e.* according to our form and Reduction,

$$\frac{a}{2} + \frac{16aa}{8} - \frac{4aa + bb}{2} \text{ i. e. } \frac{a}{2} + \frac{bb}{2} =$$

Geometrical Conſtruction. Firſt from our form, the compendiouſneſs whereof will here appear, for it requires only one preparation *n.* 2. in which LM $=$ *a*, MN and LO $=$ *b*, OP $=$ *bb*, which being premis'd, and the diameter A*y* (becauſe the quantity *p* is in the Equation) being drawn (*n.* 3.) make AI $= \frac{1}{2}$LM, and 1, 2 or 1 D $= \frac{1}{2}$OP, and DH $=$ LM. The reſt therefore being alſo perform'd, which the quantity S occurring in the preſent Equation requires, according to the laſt precepts of our introduction, you'l have NO the value of *x* ſought; whence (*n.* 4.) at the interval AB having deſcrib'd a Circle, and made a right angle at B, if FC be made $=$ to the found NO, you'l have the △ ABC required, and EC will be found at the beginning of the preſcribed magnitude.

Now if you were to find the center H by *Baker*'s form without our Reduction, 1. you muſt put of (*n.* 3.) $\frac{1}{2}$AB from A to *c*. 2. For the quantity $\frac{p^2}{8}$ make as 1 to $\frac{p}{2}$ ſo $\frac{p}{4}$ to a fourth, which would be $=$ 2AB, *viz.* 2DM, and to be ſet from *c* to *d*. 3. The quantity $\frac{q}{2}$ (to obtain which you muſt ſubſtract OP (*n.* 2.) from the quadruple of LM, and divide the remainder by 2) muſt be ſet oft backwards from *d* to *e*, by thus determining the point D. 4. The quantity *p*, which here is preciſely $=$ LM, muſt be transferr'd from D to *f* on a perpendicular erected from D. 5. For the quantity $\frac{p^3}{16}$ make as 1 to $\frac{p^2}{8}$ ($=$ 2AB) ſo $\frac{p}{2}$ $\underset{2LM}{(\text{alſo} = 2AB)}$ to a fourth quadruple of LM; and this muſt further be produc'd from *f* to the point *g* (which here the paper wil not permit) 6. For the quantity $\frac{pq}{4}$ make as 1 to $\frac{p}{2}$ ſo $\frac{q}{2}$ to a fourth, (which would be $=$ to the quadruple of LM, but taking away OP) which muſt be ſet backwards from *g* to *h*. 7. Laſtly the quantity $\frac{r}{2}$ ($=$ OP) ſet off from *h* backwards or towards the right hand to *i* will at length give the point H required.

Another

Another Solution of the ſame Problem.

This Problem may be more eaſily ſolved, and will give a far more ſimple Equation, if you are to find not FD but AE. Make therefore *(n.* 1· *Fig.* 53.) $= x$, and the reſt as above; AC will be $= x + b$, and its □ $xx + 2bx + bb$; therefore the □ BC $= xx + 2bx + bb - aa$; therefore the □ of the tangent HC $= xx + 2bbx + bb - 2aa$. But the rectangle ACE will be $bb + bx$. Therefore by *Prop.* 47. *Lib.* 1. *Math. Enucl.*

$xx + 2bx + bb - 2aa = bb + bx$;

and turning all over to the right hand,

$xx + bx - 2aa = 0$; or

$xx = -bx + 2aa$. Therefore by caſe 2 of affected quadraticks,

$x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + 2aa}$.

The *Geometrical Conſtruction* may be performed according to the rules of quadraticks *Fig.* 54. *n.* 1. as will be evident to any attentive Reader. Having therefore deſcribed a circle at the interval BA, whether it be done from any arbitrary center (ſee *n.* 4. *Fig.* 53.) or upon AE found in the preſ. *Fig.* making an interſection at the ſaid interval in B; and applying AE, and producing it until EC become equal to the given quantity *q*, and at length having drawn BC you'l have the triangle right-angled at B, and alſo the difference of the ſides FC. But to make it more ſhort and elegant; having determined AE by a little circle (*Fig* 54 *n.* 3.) prolong it to the oppoſite part of the circumference in C, and draw AB and CB; for as the radius of the little circle is $\frac{1}{2}b$, ſo EC is $= b$.

Now if you would conſtruct the Equation above by *Baker*'s rule (that its univerſality may alſo be confirm'd by an Example in quadraticks) *viz.*

$x^2 + bx - 2aa = 0$;

The centtal rule will be (taking *a* for 1 and the *L. R.*)

$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} = AD$

and

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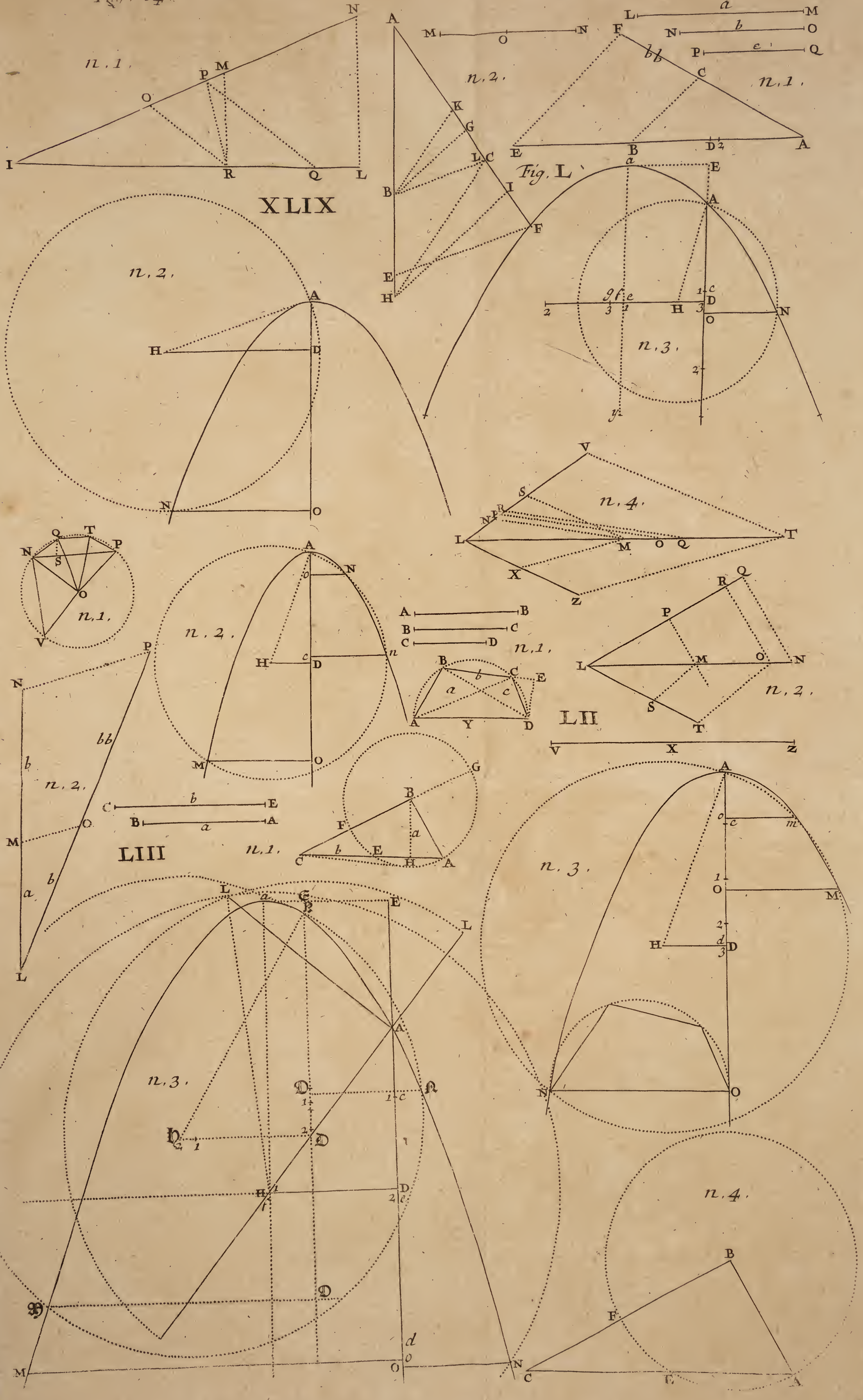

XLIX
n, 1,
N
P M
O
I R Q L
n, 2,
A
H D
N O
n, 2,
A
K
G
B L C
I
F
E
H
M O N
Fig, L
L a M
N b O
P c Q
F bb C
n, 1,
E B D 2 A
a E
A
2 3 1 H 3 D O N
n, 3,
y
n, 4,
V S R N L M O Q T X Z
n, 1,
N Q T P S O V
n, 2,
A N o H c D n M O
A B
B C
C D
n, 1,
B C E a b c A Y D
LII
R Q P L M O N S T
n, 2,
V X Z
N P bb b n, 2, M O b a L
C b E
B a A
LIII
n, 1,
G B F a C b E H A
n, 3,
A o c m 1 O M 2 H d D 3 N O
n, 4,
B F C E A
L a E E L
A
n, 3,
D 1 c N
H D
H D 2 c
D
M
d o
M O N

and $\frac{p}{4} \dagger \frac{p^3}{16} \dagger \frac{pq}{4} = DH.$ *i. e.* by our Reduction,

$$\frac{a}{2} \dagger \frac{bb}{8} \dagger aa \text{ i.e. } 1\tfrac{1}{2}a \dagger \frac{bb}{8} = AD$$

and $\frac{b}{4} \dagger \frac{b^3}{16} \dagger \frac{2aab}{4}$ i. e. $\frac{3b}{4} \dagger \frac{b^3}{16} = DH.$

The *Conſtruction* therefore will conſiſt in theſe: LM (*n.* 2. *Fig.* 54) $= a$, MO $= \frac{1}{4}b$, LP $= \frac{1}{2}b$, therefore PR $= \frac{bb}{16}:$ MN $=$ PR *viz.* $\frac{bb}{8}$, therefore Q $= \frac{b^3}{16}$. In *Fig.* 53. *n.* 3. $a\mathfrak{E} = \frac{1}{4}b$, $\mathfrak{A}\,1 = \frac{1}{2}$LM, 1, 2, or 1 $\mathfrak{D} =$ PR; $\mathfrak{D}\,1 =$ LP $\dagger$ MO, 1, 2, or 1 $\mathfrak{H} =$ PQ Having drawn a circle from $\mathfrak{H}$ thro' $\mathfrak{A}$ you'l have the true root $\mathfrak{R}\mathfrak{D} =$ to AE ſought; and $\mathfrak{M}\mathfrak{H}\mathfrak{D}$ the falſe root.

NB. Hence it is evident that one Problem may have ſeveral Solutions and Conſtructions, ſome more eaſy and ſimple, others more compound and laborious; *viz.* according as the unknown quantity is aſſumed more or leſs commodious to the purpoſe: which may not be amiſs here to note for the ſake of Learners.

PROBLEM VII.

Suppoſe a right line BD (Fig. 55.) any how divided in A, to divide it again in C, ſo that the ſquare of BA ſhall be to the ſquare of AC as AC to CD.

SOLUTION.

Since the firſt ſegments BA and AD are given, call the firſt a and the other b, and call AC the quantity AC x; and CD will be $= b - x$. Now therefore ſince we ſuppoſe

as the □ of AB to the □ AC ſo AC to CD

$$aa \text{ ——— } xx \text{ —— } x \text{ —— } b - x$$

$aab - aax$ will $= x^3$, and transferring the quantities on the left hand to the right,

$$x^3 * aax - aab = 0.$$

Wherefore the Central Rule (taking a for 1 and the *L. R.*) will be $\frac{1}{2} - \frac{q}{2} = AD$ and $\frac{r}{2} = DH$. *i. e.* by our form $\frac{a}{2} - \frac{aa}{2}$ i. e. $o = AD$, that D may fall on the vertex of the parabola;

and $\frac{aab}{2}$ i. e. $\frac{b}{2} = DH$. The Conſtruction therefore will be very ſimple, as is evident from the fifty fiſth Figure.

PROBLEM VIII.

There is given AB the capital line of a horn work (which we repreſent (tho' rudely) n. 1. *Fig.* 56.) *and the Gorge AD, alſo part of the line of defence EF, to find the face BE, the flank DE, the curtain (or the chord) DF, alſo the angle of the Baſtion ABE,* &c. *and ſo the whole delineation of the horn work.* It is evident if you have the flank DE or the curtain DF, the reſt will be had alſo. Suppoſe therefore, the capital line AB and the gorge AD, and part of the line of defence EF to be of the magnitudes denoted by the Letters a, b, c, on the right hand.

SOLUTION.

Make $AB = a$, $AD = b$, and $EF = c$, and $DF = x$; then will $AF = x \dagger b$, and by reaſon of the ſimilitude of the △ △ BAF and EDE and ECB,

as FA to AB ſo FD to DE

$$x\dagger b - a - x - \frac{ax}{x\dagger b}$$

But now □ □ DF † DE are = = □ EF i. e. $\frac{aaxx}{xx\dagger 2bx\dagger bb} \dagger xx = cc$; or giving the ſame denomination to all the quantities on the left hand,

$$\frac{x^4 \dagger 2bx^3 \dagger \begin{matrix}aa \\ \dagger bb\end{matrix} xx}{xx \dagger 2bx \dagger bb} = cc;$$

and multiplying both ſides by $xx + 2bx + bb$,

$$\begin{array}{l} x^4 + 2bx^3 + aaxx = ccxx + 2bccx + bbcc\,; \\ + bb \end{array}$$

and tranſlating all the quantities on the right hand by the contrary ſigns to the left,

$$\begin{array}{l} x^4 + 2bx^3 + aaxx - 2bccx - bbcc = o. \\ + bb \\ - cc \end{array}$$

Wherefore (putting a for 1 and *L. Rectum*) the Central Rule will be,

$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2}$ (becauſe the quantity q is negative in the Equation, for cc is greater than $aa + bb$) $=$ AD;
and $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} =$ DH. or according to our Reduction,

$$\frac{a}{2} + \frac{bb}{2} + \frac{cc - aa - bb}{2} \text{ i. e. } \frac{cc}{2} = \text{AD}$$

and $\frac{2b}{4} + \frac{8b^3}{16} + \frac{2bcc - 2aab - 2b^3}{4} - \frac{2bc}{2}$

i. e. $\frac{b}{2} + \frac{b^3}{2} + \frac{bcc - aab - b^3}{2} - bcc$ i. e.

$$\frac{o - bcc}{2} = \text{DH}.$$

Wherefore the *Geometrical Conſtruction* requires no other preparatory determination by our form than of the quantities $\frac{cc}{2}$ for AD, $\frac{bcc}{2}$ for DH at the center, and $bbcc$ to determine the radius of the circle; which are exhibited by *n.* 2. *Fig.* 56. *viz.* NP is $= cc$, RS $= bcc$, RV $= bbcc$, which are found by means of LM $= a$, LR $= b$, LN and MO $= c$, MQ $=$ NP and MT $=$ RS. Having therefore deſcribed a parabola, *n.* 3. and drawn its diameter, transfer AD $= \frac{1}{2}$ NP upon it (becauſe the quantity p is in the Equation) and alſo $\frac{1}{2}$RS from D upon H perpendicularly, and on the right hand, (becauſe DH $= -\frac{bcc}{2}$;) and ſo you'l have the center H;

thro' which having drawn KAI ſo that AK ſhall be $=$ to the quantity *bbcc* or S, *i.e.* RV, *&c.* a circle deſcribed at the interval HL will cut the parabola in M and N, and applying the magnitude NO it will be that of the Curtain ſought; upon which, *n.* 4. having laid down the circumference of the horned work by help of the given lines AB and AD, you'l have the line EF, of the magnitude which was above ſuppoſed. Now if any one has a mind to do the ſame thing by *Baker*'s way; by laying down firſt the interval AB $= \frac{L}{2}$ and then making $bc = \frac{p^2}{8}$, and laſtly, putting *cd* for the quantity $\frac{q}{2}$; he will fall upon the ſame point D, and in like manner may expreſs the other parts of the Central Rule by the intervals D*e*, *ef*, *fg*, and ſetting back the laſt *gh*, he will fall upon the ſame center H: But this is done with a great deal more trouble and labour to determine ſo many quantities, and alſo is in more danger of erring by cutting off ſo many parts ſeparately, as experience will ſhew; and thus we have by a new argument ſhewn the advantage of our Reduction.

Another Solution of the ſame Problem.

Things remaining as before (only aſſuming the given lines AB, AD and EF, *n.* 1. *Fig.* 56. one half leſs, that the Scheme may take up leſs room) make BE $= x$, as the firſt or chief unknown quantity; then will BF $= x + c$, and its $\square\ xx + 2cx + cc$: And ſince

as BE to BC $=$ AD ſo BF to AF a fourth, which will be
x —— b —— $x + c$

$= \frac{bx + bc}{x}$ and its ſquare $= \frac{bbxx + 2bbcx + bbcc}{xx}$. Wherefore if this ſquare be ſubſtracted from the ſquare of BF, there will remain the ſquare of BA, *i. e.*

$$xx + 2cx + cc \frac{- bbxx - 2bbcx - bbcc}{xx} = aa;$$ i. e. all being reduced to the ſame denomination,

$$x^4 + 2cx^3 + \begin{array}{l} ccxx \\ -bb \end{array} - 2bbcx - bbcc = aaxx;$$

or according to the forms of *Cartes* and *Baker*,

$$\begin{array}{l} x^4 + 2cx^3 + ccxx - 2bbcx - bbcc = 0. \\ \qquad\qquad\quad\; -aa \\ \qquad\qquad\quad\; -bb \end{array}$$

Therefore the Central Rule (putting again *a* for 1 and the *L. R.*) will be

$$\frac{L}{2} + \frac{p^2}{8} - \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} - \frac{pq}{4} - \frac{r}{2} = DH$;

or by our Reduction,

$$\frac{a}{2} + \frac{cc}{2} - \frac{cc + aa + bb}{2} \text{ i. e. } a + \frac{bb}{2} = AD;$$

and $\frac{c}{2} + \frac{c^3}{2} - \frac{c^3 + aac + bbc - bbc}{2}$ i. e.

$$c - \frac{bbc}{2} = DH.$$

Geometrical Conſtruction. Having therefore deſcribed a parabola (*Fig.* 56. *n.* 5.) and drawn the diameter Ay, make A1 = *a* and 1, 2 = $\frac{bb}{2}$, ſo you'l have the point D; make moreover D3 or 2, 3 = *c*, and backwards 3, 4 = $\frac{bbc}{2}$ (we here omit to expreſs the Geometrical determination of theſe quantities $\frac{bb}{2}$ and $\frac{bbc}{2}$ as being very eaſy) and you'l have the point H, &c. and there will come out the quantity ſought NO; which ſince it is equal to half BE *n.* 4. the buſineſs will be done; which *Baker*'s form will alſo give, exactly the ſame, but after a more tedious proceſs.

PROBLEM IX.

I*N any Triangle ABC (the ſcheme whereof ſee n.* 1. *Fig.* 57.) *ſuppoſe given the Perpendicular AD, and the differences of the leaſt ſide from the two others EC and FC to find all the three ſides.*

ſides. i. e. Chiefly the leaſt ſide AB which being found, the others will be ſo alſo.

SOLUTION.

Make AD $= a$, EC $= b$, and FC $= c$, AB $= x$; then will BC $= x \dagger b$ and its ſquare be $xx \dagger 2bx \dagger bb$, and AC $= x \dagger c$ and its ſquare be $xx \dagger 2cx \dagger cc$; and BD will be $\sqrt{xx - aa}$, and DC $\sqrt{xx \dagger 2cx \dagger cc - aa}$. But the □ BC may alſo be obtain'd otherwiſe, and the Equation alſo, if □ □ BD † DC † 2 ▭ of BD by DC be added into one ſum according to *Prop.* 4. *Lib.* 2. *Eucl.* viz.

$$2xx \dagger 2cx \dagger cc - 2aa \dagger \sqrt{4x^4 \dagger 8cx^3 \dagger \left.\begin{array}{r} 4cc \\ -8aa \end{array}\right\} xx - 8aacx \dagger 4a^4 - 4aacc}$$

will be $= xx \dagger 2bx \dagger bb$,

[For DC $= \sqrt{xx - aa \dagger 2cx \dagger cc}$, multiplyed by BD $\sqrt{xx - aa}$, gives the rectangle of the ſegments

$$\sqrt{x^4 \dagger 2cx^3 \dagger \begin{array}{l} ccxx \\ -2aa \end{array} - 2aacx \dagger a^4 - aacc - 2aa}$$

and this doubled *i. e.* multiplyed by $\sqrt{4}$, gives the quantity which is contain'd under the radical ſign in the Equation]

Therefore turning all over on the left hand which are before the ſign $\sqrt{}$ to the right hand, prefixing to them the contrary ſigns, you'l have

$$\sqrt{4x^4 \dagger 8x^3 \dagger \left.\begin{array}{r} 4cc \\ -8aa \end{array}\right\} xx - 8aacx \dagger 4a^4 - 4aac}$$

$$= - xx \begin{array}{l} \dagger 2b \\ - 2c \end{array} x \dagger bb - cc \dagger 2aa,$$ and taking away the *Vinculum* on the left hand, and ſquaring on the right

$$4x^4 \dagger 8cx^3 \dagger \begin{array}{r} 4cc \\ -8aa \end{array} xx - 8aacx \dagger 4a^4 - 4aacc$$

$$= x^4 \begin{array}{r} \dagger 4cx^3 \\ -4b \end{array} \left.\begin{array}{r} -4aa \\ \dagger 2bb \\ \dagger 6cc \\ -8bc \end{array}\right\} xx \left.\begin{array}{r} \dagger 4b^3 \\ \dagger 4c^3 \\ \dagger 8aab \\ -8aac \\ -4bbc \\ -4bcc \end{array}\right\} x \begin{array}{r} \dagger 4a^4 \\ \dagger b^4 \\ \dagger c^4 \\ \dagger 4aabb \\ -4aacc \\ -2bbcc \end{array}$$

and adding and ſubſtracting on both ſides, as much as can be,

$$3x^4 \dagger 4cx^3 - 4aaxx$$

$$= -b4x^3 \left.\begin{array}{r}\dagger 2bb\\ \dagger 2cc\\ -8bc\end{array}\right\} xx \left.\begin{array}{l}\dagger 4b^3\\ \dagger 4c^3\\ \dagger 8aab\\ -4bbc\\ -4bcc\end{array}\right\} x \begin{array}{l}\dagger b^4\\ \dagger c^4\\ \dagger 4aabb\\ -2bbcc\end{array}$$

and transferring all to the left,

$$3x^4 \left.\begin{array}{r}\dagger 4b\\ \dagger 4c\end{array}\right\} x^3 \left.\begin{array}{r}-4aa\\ -2bb\\ -2cc\\ \dagger 8bc\end{array}\right\} xx \left.\begin{array}{r}-4b^3\\ -4c^3\\ -8aab\\ \dagger 4bbcc\\ \dagger 4bcc\end{array}\right\} x \begin{array}{l}-b^4\\ -c^4\\ -4aabb = o\\ \dagger 2bcc\end{array}$$

and dividing all by 3,

$$x^4 \left.\begin{array}{r}\dagger \tfrac{4}{3}b\\ \dagger \tfrac{4}{3}c\end{array}\right\} x^3 \left.\begin{array}{r}-\tfrac{4}{3}aa\\ -\tfrac{2}{3}bb\\ -\tfrac{2}{3}cc\\ \dagger \tfrac{2}{3}bc\end{array}\right\} xx \left.\begin{array}{r}-\tfrac{4}{3}b^3\\ -\tfrac{4}{3}c^3\\ -2\tfrac{2}{3}aab\\ \dagger \tfrac{4}{3}bbc\\ \dagger \tfrac{4}{3}bcc\end{array}\right\} x \begin{array}{l}-\tfrac{1}{1}b^4\\ -\tfrac{1}{3}c^4\\ -\tfrac{4}{3}aabb = o.\\ \dagger \tfrac{2}{3}bbcc\end{array}$$

Note, I ſought this Equation alſo after two other ways; 1 By a compariſon of the □ AC with the two ſquares AB†BC, after 2 ▭ ▭ CBD thence ſubſtracted, according to *Prop.* 13. *Lib* 2.*Eucl* which is the 46. *Lib*.1. *Math. Enucl.* and I form'd the ſame with the preſent. 2. By putting at the beginning y for $x \dagger b$ and z for $x \dagger c$, and going on after the former methods, 'till you have this Equation,

$$x^4 \left.\begin{array}{r}-2x^2\\ \dagger 4aa\end{array}\right\} yy \left.\begin{array}{r}-2x^2\\ -2y^2\end{array}\right\} zz \begin{array}{l}\dagger y^4\\ \dagger z^4\end{array} = o.$$

in which, when afterwards I ſubſtituted the values anſwering the quantities yy and zz, *&c.* This ſame laſt Equation came out a little eaſier, but (NB) with all the contrary ſigns.

Now to form the Central Rule, and thence make the Geometrical Conſtruction, we muſt determine firſt each of the quantities p, q, r and S, that we may know whether they are negative or poſitive; and you'l find (*n.* 2. *Fig.* 57.) $p =$ G2 poſitive, $q =$ H4 negative, and K4 $= S$ alſo negative; and that by help of the quantities LM $= b$ or 1, MN and

LO $= a$, OP $= aa$, MQ and LR $= c$, RS $= cc$, and alſo LT $= cc$, TV and LX $= c^3$, XY $= c^4$. Wherefore the form of the laſt Equation will be like this,

$x^4 + px^3 - qxx - rx - S = 0$, and ſo the Central Rule (taking here b for 1 and the *L. R.*)

$$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} = AD$$

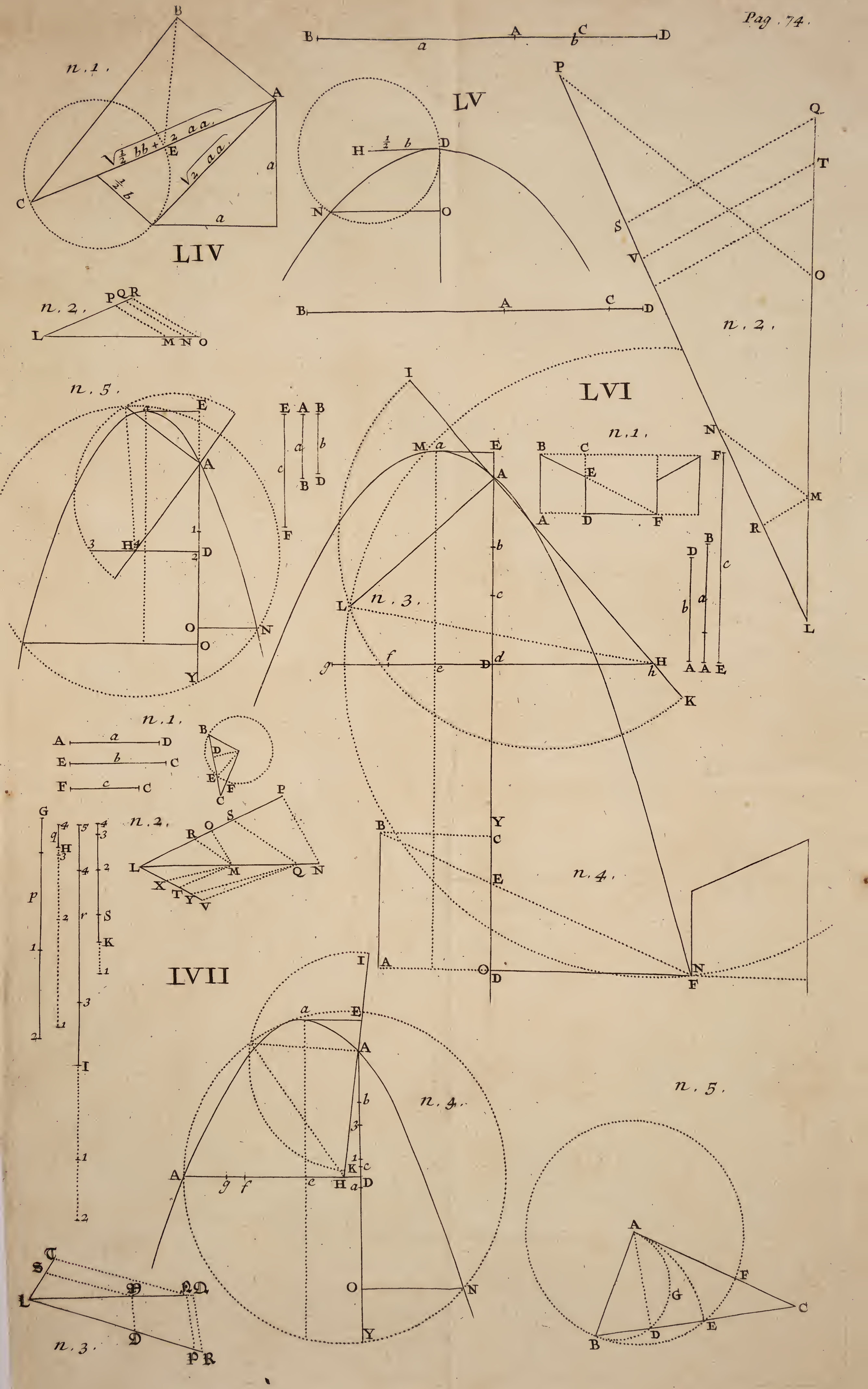
and $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} =$ DH.

Wherefore, having now deſcribed a parabola (as may be ſeen *n.* 4.) having found the diameter Ay transfer upon it firſt A$b = \frac{1}{2}$LM (*n.* 2) and then $bc = \mathfrak{DP}$ (*n.* 3.) i. e. $\frac{p^2}{8}$; and thirdly cD $= \frac{q}{2}$ i. e. $\frac{1}{2}$H4 (*n.* 2.) moreover from D to e put off $\mathfrak{MB}$ (*n.* 3. $= \frac{p}{4}$, and from e to f put off $\mathfrak{DK} = \frac{p^3}{16}$, and from f to g put off $\mathfrak{CF} = \frac{pq}{4}$; and from g backwards to h put off half the quantity r, or I5 (*n.* 2.) and having done the reſt as uſual, you'l have NO, the ſide required of the Triangle to be deſcribed; the deſcription whereof will be now eaſy (*n.* 5.) having all the three ſides known. This may ſerve for an Examen, if having deſcribed a ſemi-circle AGB upon AB $=$ NO, you apply the given line AD, and from B thro' D draw indefinitely BDC: Then at the interval AB having deſcribed the Arches AE and BF, add the given line EC to BE, for thus having joined the points A and C, FC ought to be equal to FC before given.

SCHOLIUM.

WE have here omitted our Reduction, becauſe it would be too tedious, and would expreſs the quantities AD and DH (eſpecially the latter) in terms too prolix. For AD would be $= \frac{b}{2} + \frac{5}{9}bb + \frac{5}{9}cc - \frac{8}{9}bc + \frac{2}{3}da$ (*viz.* becauſe $\frac{p^2}{8}$ is

found


LIV
n. 1.
B
A
C
E
$\sqrt{\frac{1}{2}bb+2aa}$
$\sqrt{2aa}$
$\frac{1}{2}b$
a
n. 2.
P Q R
L
M N O
LV
B a A C b D
H $\frac{1}{2}$ b D
N O
B A C D
LVI
n. 1.
B C
E
A D F
n. 2.
P
Q
T
S
V
O
N
F
M
R
L
D B
b a c
A A E
n. 3.
I
M a E
A
L
b
c
g f e D d H h
K
n. 4.
Y
B C
E
A O D N F
n. 5.
E
A
1
3 H 4 2 D
O N
O
Y
E A B
a b c
B D
F
LVII
n. 1.
A a D
E b C
F c C
B
D
E F
C
n. 2.
G
q H
p r
S
K
P
S
O
R
L M Q N
X T Y V
n. 3.
T
S
L
M N D
D
P R
n. 4.
I
a E
A
b
3
1 c
K
A g f e H a D
O N
Y
n. 5.
A
F
G
C
B D E

found $= \frac{2}{9}bb \dagger \frac{4}{9}bc \dagger \frac{2}{9}cc$ and $\frac{q}{2}$ taken in it ſelf $= \frac{2}{3}aa - \frac{1}{3}bb - \frac{1}{3}cc \dagger 43\,cc$; but here [where by vertue of the Central Rule it is taken poſitively, when it is in it ſelf negative] under contrary ſigns it is $= \frac{2}{3}aa \dagger \frac{1}{3}$ or $\frac{3}{9}bb \dagger \frac{1}{3}$ or $\frac{3}{9}cc - \frac{4}{5}$ or $\frac{12}{9}$ bc) or yet more contractedly (becauſe b is unity) AD $= \frac{1}{2}b \dagger \frac{5}{9}b$ (i. e. $1\frac{1}{18}b$) $\dagger \frac{5}{9}cc - \frac{8}{9}c \dagger \frac{2}{3}aa$; which parts may be expreſſed without any great difficulty on the Diameter Ay, by its portions A1, 1,2; 2, 3; 3D: But the other quantity DH, or the definition of the Center H, would alſo have ſome tediouſneſs, as becauſe

$\frac{p}{4}$ would be $= \frac{b \dagger c}{3}$

$$\frac{p^3}{16} = \frac{4b^3 \dagger 12bbc \dagger 12bcc \dagger 4c^3}{27}$$

$$\frac{pq}{4} = \frac{6bcc \dagger 6bbc - 4aab - 4aac - 2b^3 - 2c^3}{3}$$

or $\frac{6bbc \dagger 6bcc - 12aab - 6b^3 - 6b^3}{9}$

If you take away out of the quantities $\frac{pq}{4}$ and $\frac{r}{2}$ (ſince this latter is to be ſubſtracted, and ſo left, as it is, under the ſign $-$; but the other, alſo negative in it ſelf, but here poſitively expreſſed in the Central Rule, muſt have all the contrary ſigns) I ſay, if you take out of theſe quantities thoſe which deſtroy one another, and add the reſt with the two former quantities, they will be $\frac{b \dagger c}{3}$

$$- \frac{8aab \dagger 4aac \dagger 4bbc \dagger 4bcc}{9} - \frac{8b^3 - 8c^3}{27} = = \text{DH.}$$

or a little more contracted (becauſe b is 1) $\frac{b \dagger c}{3}$

$$- \frac{8aa \dagger 4aac \dagger 4c \dagger 4cc}{9} - \frac{8\;{}^3 - 8c^3}{27} = \text{DH.}$$

But now if any one has a mind to illuſtrate this by a numeral Example and try the truth, &c of the quantities found; they may make *e. g.* $a = 12$, $b = 1$, and $c = 2$; and they will eaſily find that in the laſt Equation the quantity p will be 4, and $q - 190$, $r - 388$, $S - 195$: Secondly in the Central Rule of *Baker* they'l find $\frac{L}{2} = \frac{1}{2}$, $\frac{p^2}{8} = 2$, and $\frac{q}{2} = 95$, and ſo the whole line AD $= 97\frac{1}{2}$; and further $\frac{p}{4} = 1$, $\frac{p^3}{16} = 4$, $\frac{pq}{4} = 190$, and $\frac{-r}{2} = -194$, and ſo the whole line DH $= 195 - 194$ *i. e.* $= 1$. Thirdly, likewiſe in our Reduction (if we proceed by each part correſponding to *Baker*'s) $\frac{b}{2} = \frac{1}{2}$, $\frac{7}{9}bb + \frac{4}{9}bc + \frac{7}{9}bc = 2$, and $\frac{1}{3}aa + \frac{4}{3}bb + \frac{4}{3}cc - 4bc = 95$. The ſum for AD $97\frac{1}{2}$; but further $\frac{b+c}{3} = 1$,

$$\frac{4b^3 + 12bbc + 12bcc + 4c^3}{27} = 4;$$

$$\frac{6bcc + 6bbc - 4aab - 2b^3 - 2c^3}{3} = 190;$$

and $2bbc + 2bcc - 4aab - 2b^3 - 2c^3 = 194$ to be ſubſtracted; and ſo the ſum for DH $= 195 - 194 = 1$. Which ſame quantities will fourthly come out, if the quantities AD and DH contracted, as they are expreſſed in letters above, be reſolved into numbers.

PROBLEM X.

YOU *are to build a Fort on the given Polygons EAF (ſee Fig. 58. n. 1.) whoſe capital line AB ſhall equal the aggregate of the gorge and flank, and the ſquares of theſe added together ſhall be equal to the ſquare of the given line GH, and the ſolid made by the multiplication of the ſquare of the flank by the gorge, ſhall be equal to the cube of the given line IK.*

SOLV-

SOLUTION.

Make the Gorge AC $= z$, whoſe ſquare zz ſubſtracted from bb the ſquare of the given line b, will leave the ſquare of the flank DC $= bb - zz$. Now this ſquare being multiplyed by the gorge AC or z will give $bbz - z^3 = g^3$, the cube of the given line IK; and adding to both ſides z^3, and ſubſtracting bbz, $o = z^3 * - bbz \dagger g^3$.

Therefore if we take g or IK for 1 and L^3, g^3 will be the line g, and

The Central Rule: $\frac{L}{2} \dagger \frac{q}{2} = AD$.

and $\frac{r}{2} = DH$.

i. e. according to our Reduction,

$\frac{g}{2} \dagger \frac{bb}{2} = AD$ and $\frac{g}{2}$ DH.

Geometrical Conſtruction. Having deſcribed a Parabola (*n.* 2. *Fig.* 58.) make on its ax A1 $= \frac{1}{2}$IK and 1, 2, ſ. *viz.* 1D $= \frac{1}{2}$MN (from *n.* 1.) and DH $= \frac{1}{2}$IK. Then having deſcribed a circle from H, and found the true root NO upon the given angle EAF (*n* 3.) make AC $=$ NO, and having erected the perpendicular CD divide it by AD $=$ GH (*n.* 1.) and make AB $=$ AC † CD; and the Fort will be drawn.

PROBLEM XI.

IN a right-angled Triangle ABC (which we denote by n. 1. Fig. 59.) having given the greater ſide AC, and made the leſs ſide AB = to the ſegment CE, which ſhall cut off from the baſe BC a perpendicular let fall from the right angle A; to find theſe lines AB or CE, and conſequently the whole triangle.

SOLUTION.

Make AC $= a$, CE or AB $= x$. Therefore, 1. you'l have $aa - xx = \square$ AE. And becauſe the $\triangle \triangle$ BEA and CAE are ſimilar, you'l have

as AC to CE ſo BA to AE

$$a \text{——} x \text{——} x \text{——} \frac{xx}{a}$$

And ſo,2. $\square$ AE $= \frac{x^4}{aa}$. Therefore

$$\frac{x^4}{aa} = aa - xx;$$ and multipl. by *aa*,

$x^4 = a^4 - aaxx$; or, according to the form of *Cartes* and *Baker*, tranſpoſing all to the left,

$x^4 * + aaxx * - a^4 = o$ i. e.

$x^4 * + qxx * - S = o$.

Therefore (taking a for 1 and *L. R.*) the Central Rule will be $\frac{L}{2} - \frac{q}{2}$ i. e. $o =$ AD

and $\frac{r}{2}$ i. e. $o =$ DH; that H may fall on the vertex A.

Geometrical Conſtruction. Since a is aſſumed for unity and L, the quantity S alſo and *Latus Rectum* i.e. AI and AK and conſequently the mean proportional AL and the radius HL will be = = to the given ſide AC, and conſequently at that interval having deſcribed a circle, thro' the Parabola rightly delineated, you'l have NO the value of the quantity x, *i. e.* of the leſſer ſide AB. Having drawn therefore NA, which is = to AC by Conſtruction, if you draw to it the perpendicular AB cutting NO produc'd to B, you'l have the Triangle ſought ABC, and AB (which will be a ſign of a true Solution) will be found = NO or CE.

Another Conſtruction. Since in the Equation above found there is neither x^3 nor x, it may be look'd upon as a quadratick, and conſtructed after the ſame way, as ſeveral other like it among the Examples *n.* 4. *viz.* becauſe $x^4 = -aaxx + a^4$; according to caſe 2 of affected quadraticks,

$$xx = -\tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + a^4} \text{ i. e. } + \sqrt{\tfrac{5}{4}a^4},$$

and $x = \sqrt{-\frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}}$ i. e. $\sqrt{-\frac{1}{2}a + \sqrt{\frac{5}{4}a}}$ Wherefore (*n.* 3. *Fig.* 59.) if AC be made $= a$ and CD $\frac{5}{4}a$, the mean proportional CG will be $= \sqrt{\frac{5}{4}a}$, and taking hence GF $= \frac{1}{2}a$, there will remain FC $= \sqrt{\frac{5}{4}a} - \frac{1}{2}a$: And now between this or CH equal to it, and unity AC, having found another mean proportional CE it will be the value of the quantity x ſought $=$ NO (*n.* 2.)

PROBLEM XII.

IN a right-angled Triangle ABC (Fig. 60. *n.* 1.) *there is given the Perpendicular BA, a ſegment of the Hypothenuſa BD, and a ſegment of the Baſe EC, from C to the perpendicular DE let fall from the end of the ſegment BD; to find AE DC, and conſequently the Baſe AC and the Hypothenuſa BC, and ſo the whole Triangle.*

SOLUTION.

Make AB $= a$, BD $= b$, and EC $= c$, and DC $= x$; which being given, the reſt cannot be wanting: Therefore $xx - cc = \square$ DE. But the ſame $\square$ DE may be had, if you infer

as BC to BA ſo DC to DE

$$b+x - a - x - \frac{ax}{b+x}$$

And then ſquare the quantity DE, the ſquare will be

$$\frac{aaxx}{xx+2bx+bb} = xx - cc;$$

and multiplying both ſides by $xx + 2bx + bb$,

$$aaxx = x^4 + 2bx^3 + \underset{-cc}{bbxx} - 2bccx - bbcc;$$

and ſubſtracting alſo $aaxx$,

$$x^4 + 2bx + \underset{\substack{-aa\\-cc}}{bbxx} - 2bccx - bbcc = 0.$$

Which Equation (ſince *bb* is greater than *aa* and *cc* taken together) will give this form according to *Baker*,

$$x^4 + px^3 + qxx - r - S = o.$$

Wherefore the Central Rule will be (taking *a* for unity and *L*.)

$$\frac{L}{2} + \frac{p^2}{8} - \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} - \frac{pq}{4} - \frac{r}{2} = DH$;

or according to our Reduction,

$$\frac{a}{2} + \frac{bb}{2} - \frac{bb + aa + cc}{2} \text{ i. e. } \frac{a + cc}{2} = AD;$$

and $\frac{b}{2} + \frac{b^3}{2} - \frac{b^3 + aab + bcc - bcc}{2}$ i. e.

$$b - \frac{bcc}{2} = DH.$$

Geometrcal Conſtruction, for the foundation of which there are found (*n.* 2.) and partly aſſumed, LM = *a*, MN and LO = *c*, OP and MQ = *cc*, LR = $\frac{1}{2}b$, RS = $\frac{1}{2}bcc$; (*n* 3.) $\mathfrak{LM} = a$, $\mathfrak{LN}$ and $\mathfrak{LO} = b$, $\mathfrak{LP} = bb$, $\mathfrak{LQ} = cc$, $\mathfrak{LR} = bbcc$. Having therefore delineated the Parabola, (*n* 4.) and having found its diameter, put thereon LM from A to 1, and $\frac{1}{2}$OP from 1 to 2 or D; and moreover BD (*n.* 1.) from D to 3, and RS backward from 3 to 4 or H, the Centre ſought. Then having made AK = $\mathfrak{LR}$, and done the reſt as uſual, you'l find NO or the quantity DC for the value of *x*; to which if you add BD and, having upon the whole line BC deſcribed a ſemi circle, apply in it BA, and you'l have alſo AC and the whole Triangle ſought, and you'l find EC (letting fall a Perpendicular from D) equal to that quantity which had the ſame name *n.* 1.

If any one had rather immediately conſtruct *Baker*'s form, there will be more to be prepar'd and aſſum'd; *viz.* (*n* 5.) $\mathfrak{LP} = bb$, $\mathfrak{LM} = a$ or *aa*, $\mathfrak{MQ} = cc$, or OP (*n* 2.) that you may have $\mathfrak{QP} = q$: Moreover PQ = *a*, PR = $\frac{1}{2}p$ or BD (*n.* 1.) QS = $\frac{1}{4}p$ or $\frac{1}{2}$BD; that RT may be $\frac{pp}{8}$ and alſo PV $=$ to it, that PX may be $\frac{p^3}{16}$; and laſtly P*y* $= \frac{1}{2}q$, that PZ may be $\frac{pq}{4}$. Theſe being thus prepar'd, if (*n.* 4) A*b* be made $= \frac{1}{2}$ AB (*n.* 1.) *bc* $=$ RT, and *cd* backwards $= \frac{1}{2}$ 𝔓𝔇 we ſhall light on the point D; and, if we make D*e* $= \frac{1}{2}$ BD, *ef* $=$ PX (*n.* 5. which interval was too big to be repreſented in the Paper) and from *f* you put backwards *fg* $=$ PZ, and on from *g* beyond to the right hand *gh* $=$ 2RS; we ſhall light on the point H, *&c.* Which of theſe two methods is the ſhorteſt and fitteſt for practice, any one, never ſo little experienc'd, may here ſee; and firſt Learners may take notice if they would conſtruct by *Baker*'s form, in the Diagrams *n.* 3. *n.* 5. and the like, they muſt take care to make the angles 𝔓𝔏𝔑, SPT pretty large; which we have here repreſented the leſs to ſave charges in cutting on Copper.

PROBLEM XIII.

Having given the Diameter of a Circle CD (n. 1. *Fig.* 61.) *and the line BG, which falls on it perpendicularly, (which we have here only rudely delineated) to find the point A. from which a right line AC being drawn ſhall ſo cut the line BG in F, that AF, FG, GD ſhall be three continual Proportionals.*

SOLUTION.

If CF be found, the point A will be alſo had, and the ſection of the line BG will be made. Make therefore CF $= x$, and (becauſe the perpendicular BG is given, there will alſo be given the ſegments of the diameter CG and GD) make GD $= b$, and CG $= c$; then will BG $= \sqrt{bc}$ and CD $= b + c$. Since therefore the △ △ CAD and CGF are right-angled, and have the angle at C common, they will be ſimilar.

Therefore, as CF to CG ſo CD to CA

$$x \text{ --- } c \text{ --- } b+c \text{ --- } \frac{bc+cc}{x}$$

Therefore AF $= \frac{bc+cc}{x} - x$, and FG $\sqrt{xx-cc}$.

But by the Hypotheſis,

as AF to FG ſo FG to GD

$$\frac{bc+cc-xx}{x} \;\text{—}\; \sqrt{xx-cc} \;\text{—}\; \sqrt{xx-cc} \;\text{—}\; b.$$

Therefore the rectangle of the extremes will be equal to that of the means,

$$\frac{bbc+bcc-bxx}{x} = xx-cc;$$

and multiplying by x,

$$bbc+bcc-bxx = x^3-ccx;$$

and tranſpoſing all to the left,

$$x^3+bxx-ccx-bbc-bcc = o;$$ i. e. by the *Carteſian* form,

$$x^3+pxx-qx-r = o.$$

Therefore (taking b for 1 and *L*.) the Central Rule will be,

$$\frac{L}{2}+\frac{p^2}{8}+\frac{q}{2} = AD$$

and $\frac{p}{4}+\frac{p^3}{16}+\frac{pq}{4}-\frac{r}{2} =$ DH.

or according to our Reduction,

$\frac{b}{2}+\frac{bb}{8}+\frac{cc}{2}$ i. e. $\frac{5b}{8}+\frac{cc}{2} =$ AD

and $\frac{b}{4}+\frac{b^3}{16}+\frac{cc}{4}-\frac{c-cc}{2}$ i. e. $\frac{5b}{16}-\frac{cc}{4}-\frac{c}{2} =$ DH.

Geometrical Conſtruction. Having deſcribed upon the given line CD (*n* 2) a ſemi-circle, and apply'd in it the given perpendicular BG, as the figure ſhews, you'l have the ſegments of the Diameter GD $= b$, and to the quantity p in *Baker*'s form, and CG $= c$, which (*n*. 3 where LM $= b$, LO and MN $= c$) will give OP $= cc$ and to the quantity q. Wherefore having deſcrib'd a Parabola (*n*. 4.) and the line VZ $= 2\frac{1}{2}b$, having cut off the fourth part of XZ, and the eighth of YZ (whereof the one will be $= \frac{2\frac{1}{2}b}{8}$, viz. $\frac{5b}{16}$, and the other to $\frac{5b)}{8}$ if you transfer A1 = XZ upon the diameter of the Parabola A*y*, and moreover 1, 2 or 1 D = to half OP (*n*.3.) and tranſverſly D3 = YZ and backwards 3, 4 = $\frac{1}{4}$ OP, as alſo 4, 5 = $\frac{1}{2}$CG (*n*. 2.) you'l have the center H, and having deſcrib'd a Circle at the interval HA, the root NO muſt be transferr'd from (*n*. 2.) C to F, and continued to A the point ſought. In *Baker*'s Form (becauſe the quantity *p* is = *b* or 1) $\frac{p^2}{8}$ is = $\frac{b}{8}$ and $\frac{b^3}{16}$ = $\frac{b}{16}$, and the quantity *q* or *cc* = OP, (*n*. 3.) make therefore in the Diameter of the Parabola A*b* = $\frac{1}{2}$GD, and *bc* = $\frac{1}{8}$GD, (*n*. 2.) and laſtly *cd* = $\frac{1}{2}$OP (*n*. 3.) and you'l have the point D the ſame as before. Make moreover D*e* = $\frac{1}{4}$GD and *cf* = $\frac{1}{16}$CG, and then backwards *fg* = $\frac{1}{4}$OP, and laſtly *gh* = $\frac{1}{2}$GD, and you'l have the ſame center H, and the coincidence of the parts in both forms will be pleaſant to obſerve; which otherwiſe ſeldom happens.

Other Solutions of the ſame Problem.

Carolus Renaldinus, from whoſe Treatiſe *de Reſol. & Compoſ. Math. Lib.* 2. we have the preſent Problem, proceeds to ſolve it in another way, changing it plainly into another Problem: *viz.* he obſerves, 1. That the angles FAD (ſee *n*. 1. of our 61. *Fig.*) and FGD, ſince both are right ones on the ſame common baſe FD, are in circle. Hence he infers, 2. (by vertue of the *Coroll.* of the 26. *Prop.* 3. *Eucl.*) that the ▭ ▭ DCG and ACF are equal, and conſequently CD, CA, FC and CG are four continued proportionals. Then he obſerves, 3. That GD is the exceſs of the firſt of theſe proportionals above the fourth CG, and AF is the exceſs of the ſecond AC above the third CF; and ſo, ſince 4. the rectangle of AF and GD is = to the ſquare of the mean proportional FG (for AF, FG, GD, are ſuppoſed to be continual proportionals) and this □ FG is the exceſs, by which the ſquare of the third CF exceeds the ſquare of the fourth CG; now the preſent Problem will be 5. reduc'd to this other: Having two right lines (CD and CG) given to find two ſuch mean proportionals (AC and FC) that the ▭ of the exceſs of the firſt above the fourth (*viz.* of FA into GD) ſhall be equal to the exceſs, by which the ſquare of the third (FC) exceeds the ſquare of the fourth (CG) *viz.* by the ſquare FG.

Wherefore inſtead of the former he ſolves this latter Problem, putting for CG, b, for GD, c, ſo that the firſt of the given lines CD ſhall be $= b + c$, and the other GD $= b$; calling the firſt mean proportional AC, x; and thence denominating the latter $\frac{bb + bc}{x}$ (*viz.* multiplying the fourth by the firſt, and dividing the product by the ſecond) and moreover he finds the exceſs of the firſt $(b + c)$ above the fourth (b) to be c, and the exceſs of the ſecond (x) above the third $\left(\frac{bb + bc}{x}\right)$ to be $x - \frac{bb + bc}{x}$ i. e. $\frac{xx - bb + bc}{x}$; ſo that the ▭▭ of theſe two exceſſes is $\frac{cxx - bbc + bcc}{x}$, and becauſe the $\square$ of the third FC is $= \frac{b^4 + 2b^3c + bbcc}{xx}$, having ſubſtracted $bb = \square$ GC, there is given the $\square$ FG $= \frac{b^4 + 2b^3c + bb}{x} - bb =$ ▭ of the exceſſes we juſt now found. So that now you'l have the Equation

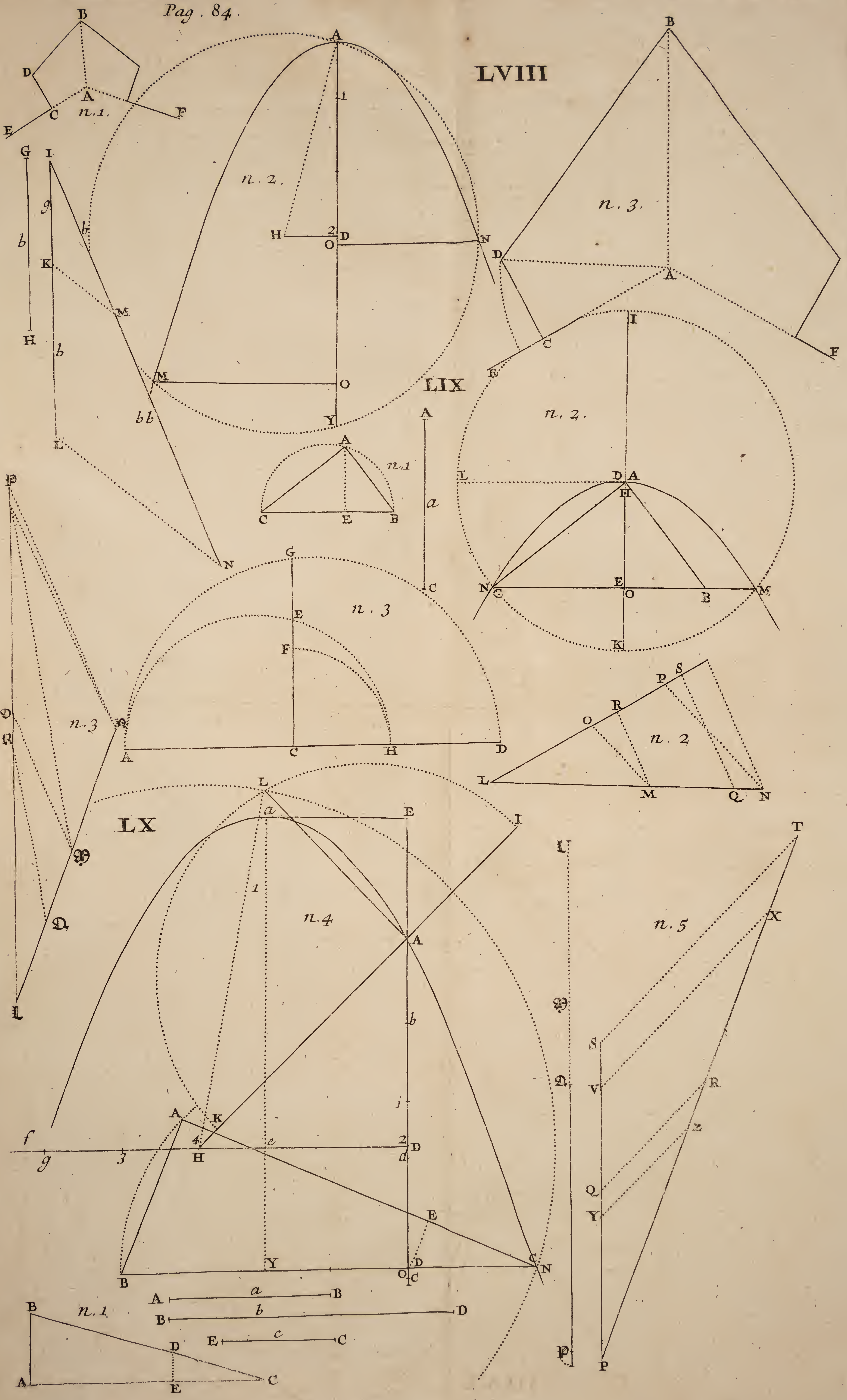
$$\frac{b^4 + 2b^3c + bbcc - bbxx}{xx} = \frac{cxx - bbc + bcc}{x}, \text{ \&c.}$$

We alſo endeavour'd to find another Solution, by finding an Equation from the line BD (*n.* 1. *Fig.* 61.) as which might be twice obtain'd by means of the two right-angled Triangles FAD and FGD, ſince it is the hypothenuſa of both. But here, beſides the former denominations of our Solution we muſt firſt give a denomination to the line AD, by making

as CF to FG ſo CD to AD

$$x - \sqrt{xx - cc} - b + c \text{ ſo } a - \frac{a\sqrt{xx - cc}}{x}, \text{ \&c.}$$

But whoſoever ſhall proſecute this Solution of ours, or that of *Renaldinus* to the end, will find much more labour and difficulty in either, than in the firſt we have given.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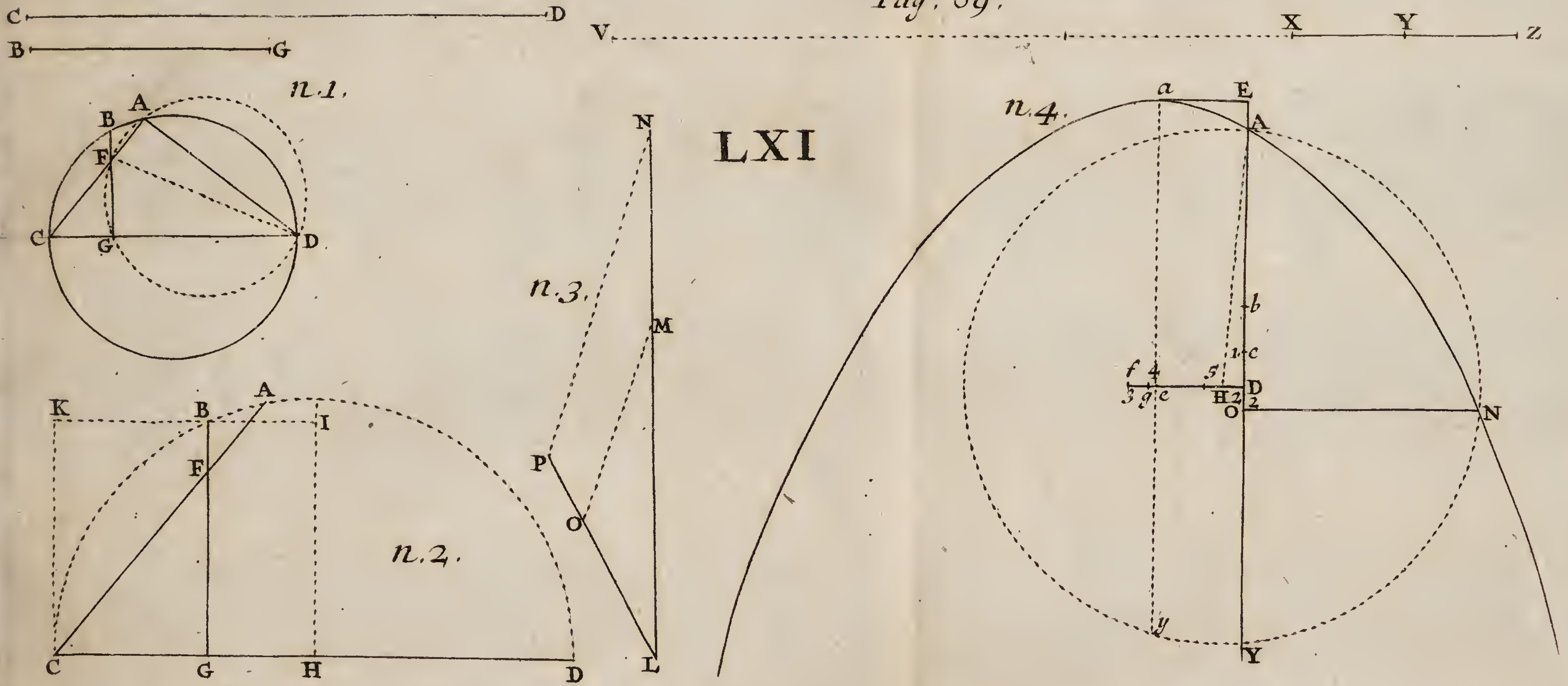
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APPENDIX.

THE Invention of the ſpecial Central Rule for the caſe of *Problem* 1. *Of Cubick Equations*, &c. which may ſerve for an Example for all the other ſpecial ones which belong to our Synopſis, *p.* 354. (and from theſe ſpecial ones) to find a general one.

In *Fig.* 47. *n.* 4. make AD $= b$, DH $= d$; and ſo we ſhall have the $\square$ of the radius HA $= bb + dd$. But this $\square$ HA or HN, may be alſo had otherwiſe, by putting,

2. For the quantity NO, as ſought, the letter x, and by inferring from the known Property of the Parabola,

as L to NO ſo NO to AO

$$—x— \quad x \quad — \frac{xx}{L}$$

and ſubſtracting AD $= b$ from AO, you'l have DO or PN $= \frac{xx}{L} - b$; whoſe $\square$ is $\frac{x^4}{L^2} - \frac{2bxx}{L} + bb$.

But you alſo have PH $=$ DH — DP or NO, *i. e.* $d - x$; whoſe $\square$ is therefore $= dd - 2dx + xx$.

Wherefore adding the $\square$ $\square$ PN and PH, you'l have the $\square$ HN $= \frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx + bb + dd$.

Wherefore the Equation will now be readily had:

$$(\square \text{ HN}) \text{——————————} (\square \text{ HA})$$

$$\frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx + bb + dd = bb + dd;$$

and taking from both ſides $bb + dd$,

$$\frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx = 0.$$

and multiplying all by L^2
and alſo dividing by x,

$$\begin{array}{lll} x^3 \; * & - 2bLx & - 2L^2d = 0. \\ & + L^2x & \end{array}$$

3. And now farther comparing this Equation with a form like ours in *Probl.* 1. *viz* with this,

$$x^3 * * — r \boxminus 0;$$

It is manifeſt, ſince in ours alſo the third term, *viz.* q is wanting, that the correſpondent one to it in the former $— 2bL \dagger L^2$ is equivalent to 0; and adding to both ſides $2bL$, L^2 will be made $\boxminus 2bL$; and dividing by $2L$, $\frac{L^2}{2L}$ will $\boxminus b$ or AD. In like manner, ſince $— r$ in our form anſwers to the quantity $2L^2d$ in the former; $2L^2d$ will $\boxminus r$, and, dividing by $2L^2$, DH or d will $\boxminus \frac{r}{2L^2}$: Which is the other Member of the Central Rule to be found.

NB. The Analytick Art has this particularly to be admired in in, that it finds its own Rules by an Analyſis: Whereof we have here an evident Example, and ſeveral others in the Reſolution of affected Quadraticks, *p.* 345. and the following.

II.

The Invention of the Central Rule, in the Caſe of Fig. 11. *and the like.*

1. $\square$ HA $\boxminus bb \dagger dd$ as above.

2. Putting x for NO, as ſought, we may infer from a new Property of the Parabola, which we have demonſtrated *Prop.* 6. *lib.* 2.

as L to NO ſo OR to AO, *i. e.* (putting a for BA or FO given; that NO — OF *i. e.* NF or OR ſhall be $\boxminus x — a$)

as L to x ſo $x — a$ to $\frac{xx — ax}{L} \boxminus$ AO.

Therefore having ſubſtracted AD *i. e.* b from AO, you'l have DO or HP $\boxminus \frac{xx — ax — b}{L}$; whoſe $\square$ is

$x^4 —$

$$\frac{x^4 - 2ax^3 \dagger axx}{L} - \frac{2bxx}{L} \dagger \frac{2abx}{L} \dagger bb.$$

But PN alſo is ⊐ NO — PO or HD, ⊐ $x - d$, whoſe □ is ⊐ $x^2 - 2dx \dagger dd$.

Therefore having added the □ □ PH and PN, there will come out □ HN

$$\frac{x^4 - 2ax^3 \dagger axx}{L^2} - \frac{2bxx \dagger 2abx}{L} \dagger xx - 2dx \dagger bb \dagger dd,$$

⊐ □ HA *i. i.* $bb \dagger dd$; and taking away from both ſides $bb \dagger dd$,

$$\frac{x^4 - 2ax^3 \dagger axx}{L^2} - \frac{2bxx \dagger 2abx}{L} \dagger xx - 2dx \sqsupset 0;$$

and multiplying every where by L^2 and diving by x,

$$x^3 - 2ax^2 \dagger aax - 2Lbx \dagger 2Lab \dagger 2L^2x - 2L^2d \sqsupset 0.$$

3. And now comparing this Equation with another form, which ſhall be like an Equation ariſing from the Solution of ſome Problem, *e. g.* with this $x^3 - pxx \dagger qx - r \sqsupset 0$; to this you'l have ⊐ this other,

$$x^3 - 2axx \left.\begin{matrix} \dagger\, aa \\ -2Lb \\ \dagger\, L^2 \end{matrix}\right\} x \begin{matrix} \dagger\, 2Lab \\ -2L^2d. \end{matrix}$$

4. Wherefore, becauſe in theſe equal forms, firſt, $2a$ is ⊐ p, a will be ⊐ $\frac{p}{2}$ *i. e.* the line BA. Secondly, Becauſe aa (or $\frac{p^2}{4}$) $\dagger L^2 - 2Lb \sqsupset q$;

Therefore $2Lb \sqsupset \frac{p^2}{4} \dagger L^2 - q$, and dividing by 2L,

$$b \sqsupset \frac{L^2}{2L} \text{ (or } \frac{L}{2}\text{)} \dagger \frac{p^2}{8L} - \frac{q}{2L} \sqsupset \text{AD}.$$

Thirdly, Becauſe $2abL - 2dL^2 \sqsupset r$ *i. e.*
$2abL - r \sqsupset 2dL^2$; therefore dividing by $2L^2$, d will be ⊐ $\frac{2abL}{2L^2} - \frac{r}{2L^2}$ i. e.

$d = \frac{ab}{L} - \frac{r}{2L^2}$ *i. e.* reſolving $\frac{ab}{L}$ into equivalent Terms expreſſed by p and q,

$d = \frac{p}{4} + \frac{p^3}{16L^2} - \frac{pq}{4L^2} - \frac{r}{2L^2}$; which is the other member of the Central Rule to be found.

viz. a is $= \frac{p}{2}$; $b = \frac{L}{2} + \frac{p^3}{8L} - \frac{q}{2L}$

Therefore ab will be $= \frac{Lp}{4} + \frac{p^3}{16L} - \frac{pq}{4L}$

Therefore $\frac{ab}{L} = \frac{p}{4} + \frac{p^3}{16L^2} - \frac{pq}{4L^2}$.

THE

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